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Reliability Estimation of n-Cascade System for Inverse Exponential Distribution

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Abstract

This paper presents the estimation of reliability for n-cascade stress-strength model. In this paper, all the components are assumed to be independent and follow one-parameter inverse exponential distribution. Reliability is estimated by using maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE), minimum mean squared error estimator (MinMse). Numerical computations have been done for reliability by using simulation techniques.

Key words: Reliability, Inverse Exponential distribution, MLE, UMVUE, MINMSE, n-cascade system**.**

1. Introduction

Cascade system is a special type of cold standby system. Cascade redundancy is a standby redundancy where a standby component taking the place of the failed component is subjected to a changed value of the stress. The n-cascade system reliability is defined by Sriwastav (1975) [1]. Raghavachar et al., (1983) [2] who derived an n-cascade system when the strengths of the various components and the imminent stresses on the system are random and follow non-identical probability distributions. Pandit et al., (1986) have derived an expression for stress-strength models of an n-cascade system when the components on the system are random and follow exponential distribution. T.S.Uma Maheshwari et al., (1993) [3] studied the reliability comparison of n-cascade system with addition of n-strengths system. Rekha and Shyam Sundar (1997) [4] studied the reliability of a cascade system with exponential stress and gamma stress. The Maximum likelihood estimation method was proposed by R.A. Fisher (1920) and has been widely used. This method is the most popular procedure for estimating the unknown parameter ϴ which specifies a probability function f(x;ϴ) , based on the observations ($x_1, x_2, x_3, \ldots, x_n$). The Uniform Minimum Variance Unbiased Estimator was first obtained by Aitken and Silverstone (1942) in the situation in which the information inequality yields the same result. In this paper the UMVUE of the scale parameter (ϴ) for the one parameter Inverse Exponential distribution has been obtained. The estimator based on minimum mean square error of the fitted values of a dependent variable, which is a typical measure of estimator quality (Drastik, 1984) [5]. The Mean Square Error (MSE) of an estimator is the difference between values implied by an estimator and the observed values of the quantity being estimated, or the average of the squares of the errors. The error is the amount by which the value implied by the estimator differs from the quantity to be estimated.

2. Statistical Model:

If the independent random variables X and Y represent strength and stress respectively, then reliability for stress-strength model is given by

$$R=P(X>Y) \;\ldots\ldots\; (1)$$

If the p.d.f of X and Y are f(x) and g(y) respectively,

$$R=\int_0^{\infty}\left(\int_y^{\infty} f(x)dx\right)g(y)dy$$

Then $F(y)=\int_0^{y} f(x)dx$

$$R = \int_{-\infty}^{\infty} \overline{F}(y)\, g(y)\, dy$$

The reliability of n-cascade system could survive with a loss of the first (n-1) components if and only if $X_i \le Y_i$;i=1,2,3,…,n-1 and $X_n > Y_n$.

The system reliability of n-components for cascade model, R_n, of the system is given by.

$$R_n = \sum_{i=1}^{n} R(i)$$

$$R(n) = P[X_1 \le Y_1, X_2 \le Y_2, ..., X_{n-1} \le Y_{n-1}, X_n > Y_n]$$

$$R(n) = P\left[\left\{\bigcap_{i=1}^{n-1}(X_i \le Y_i)\right\}\bigcap(X_n > Y_n)\right] \quad ……………..(2)$$

In cascade system, after every failure the stress is modified by a factor k which is called “attenuation factor” .so $Y_2 = kY_1$, $Y_3 = kY_2 = k^2Y_1$,………..,$Y_i = k^{i-1}Y_1$,i=1,2,3,…,n .

From equation (2)

$$R(n) = P\left[\left\{\bigcap_{i=1}^{n-1}(X_i \le k_i^{i-1}Y_i)\right\}\bigcap(X_n > k_n^{n-1}Y_1)\right]$$

$$R(n) = \int_0^{\infty}\left[\int_0^{y_1} f_1(x_1)dx_1 \mathrm{X} \int_0^{ky_1} f_2(x_2)dx_2 \mathrm{X} .... \mathrm{X} \int_0^{k^{n-2}y_1} f_{n-1}(x_{n-1})dx_{n-1} \mathrm{X} \int_{k^{n-1}y_1}^{\infty} f_n(x_n)dx_n\right] g(y_1)dy_1$$

$$R(n) = \int_0^{\infty}\left[F_1(y_1)F_2(ky_1)..........F_{n-1}(k^{i-1}y_1)\overline{F}_n(k^i y_1)\right] g(y_1)dy_1$$

$$F_n(k^i y_1) = \int_0^{k^i y_1} f_i(x_i)dx_i$$

$$\overline{F}_n(k^i y_1) = 1 - F_n(k^i y_1)$$

2.1 Stress-Strength follow inverse exponential distribution:

Let us suppose that X is strength variable with probability density function f(x) and cumulative distribution function F(x)

$$f_i(x_i, \lambda) = \frac{\lambda_i}{x_i^{2}} \exp\left(-\frac{\lambda_i}{x_i}\right), x_i \ge 0, \lambda_i > 0$$

$$F_i(x_i, \lambda_i) = \exp\left(-\frac{\lambda_i}{x_i}\right), x_i \ge 0, \lambda_i > 0$$

And Y is stress variable with probability density function g(y) and cumulative distribution function G(y)

$$g(y_i, \alpha) = \frac{\alpha}{y_i^{2}} \exp\left(-\frac{\alpha}{y_i}\right), y_i \ge 0, \alpha > 0$$

Reliability for cascade system

R (1) =P ($X_1 \ge Y_1$)

$$= \int_0^{\infty}\left[\int_{y_1}^{\infty} f(x_1)dx_1\right] g\ (y_1)dy_1$$

$$= \int_0^\infty \overline{F_1(y_1)} g\ (y_1) dy_1$$

$$\overline{F_1(y_1)} = 1 - F_1(y_1)$$

$$R\ (1) = \int_0^\infty [1 - F_1(y_1)] g\ (y_1) dy_1$$

$$R(1) = \int_0^\infty \left[1 - e^{\left(-\frac{\lambda_1}{x_1}\right)}\right] \frac{\alpha}{y_1^2} e^{\left(\frac{-\alpha}{y_1}\right)} dy_1$$

$$R(1) = \frac{\lambda_1}{\lambda_1 + \alpha}$$

R (2)=P($X_1<Y_1, X_2 \geq kY_1$)

$$R(2) = \int_0^\infty F_1(y_1)[1 - F_2(ky_1)] g(y_1) dy_1$$

$$R(2) = \frac{\alpha}{\lambda_1 + \alpha} - \frac{k\alpha}{k(\lambda_1 + \alpha) + \lambda_2}$$

$$R(3) = P(X_1 < Y_1, X_2 < kY_1, X_3 \geq K^2 Y_1)$$

$$R(3) = \int_0^\infty F_1(y_1) F_2(ky_2)[1 - F_3(k^2 y_1)] g(y_1) dy_1$$

$$R(3) = \frac{k\alpha}{k(\lambda_1 + \alpha) + \lambda_2} - \frac{k^2\alpha}{k^2(\lambda_1 + \alpha) + k\lambda_2 + \lambda_3}$$

$$R(4) = P(X_1 < Y_1, X_2 < kY_1, X_3 < K^2 Y_1, X_4 \geq k^3 Y_1)$$

$$R(4) = \int_0^\infty F_1(y_1) F_2(ky_2) F_3(k^2 y_3)[1 - F_4(k^3 y_1)] g(y_1) dy_1$$

$$R(4) = \frac{k^2\alpha}{k^2(\lambda_1 + \alpha) + k\lambda_2 + \lambda_3} - \frac{k^3\alpha}{k^3(\lambda_1 + \alpha) + k^2\lambda_2 + k\lambda_3 + \lambda_4}$$

$$R(n) = \frac{k^{n-2}\alpha}{k^{n-2}(\lambda_1 + \alpha) + k^{n-3}\ \lambda_2 + \ldots\ldots + \lambda_n} - \frac{k^{n-1}\alpha}{k^{n-1}(\lambda_1 + \alpha) + k^{n-2}\lambda_2 + k^{n-3}\lambda_3 + \ldots\ldots + \lambda_n}, n \geq 2$$

3.1 Maximum Likelihood Estimation (MLE)

If $x_1, x_2, \ldots, x_n$ is a random sample of f(x) and $y_1, y_2, \ldots, y_n$ is a random sample of g(y) then the likelihood estimation is given by

$$L(\theta; x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta)$$

$$L(\lambda_i, \alpha) = \prod_{i=1}^{n} \frac{\lambda_i}{x^2} \exp\left(-\frac{\lambda_i}{x}\right) \frac{\alpha}{y^2} \exp\left(-\frac{\alpha}{y}\right)$$

$$LogL = n\log(\lambda_i) - 2\sum_{i=1}^{n} \log x_i - \lambda_i \sum_{i=1}^{n} \frac{1}{x_i} + n\log(\alpha) - 2\sum_{i=1}^{n} \log y_i - \alpha \sum_{i=1}^{n} \frac{1}{y_i}$$

$$\frac{\partial \log L}{\partial \lambda_i} = \frac{n}{\lambda_i} - \sum_{i=1}^{n} \frac{1}{x_i} = 0$$

$$\hat{\lambda}_i = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}, i = 1,2,\ldots,n$$

similarly

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \frac{1}{y_i}}$$

Reliability estimation of 1-cascade system of type (X>Y) is

$$\hat{R}(1) = \frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \hat{\alpha}}$$

Reliability estimation of n- cascade system of type(X>Y) is

$$\hat{R}(n) = \frac{k^{n-2}\hat{\alpha}}{k^{n-2}(\hat{\lambda}_1 + \hat{\alpha}) + k^{n-3}\ \hat{\lambda}_2 + \ldots\ldots + \hat{\lambda}_n} - \frac{k^{n-1}\hat{\alpha}}{k^{n-1}(\hat{\lambda}_1 + \hat{\alpha}) + k^{n-2}\hat{\lambda}_2 + k^{n-3}\hat{\lambda}_3 + \ldots\ldots + \hat{\lambda}_n}, n \geq 2$$

3.2 Uniformly Minimum Variance Unbiased Estimators (UMVUE)

Let T be the Uniformly Minimum Variance Unbiased estimator (UMVUE) for one parameter Inverse exponential distribution for the random variable X, then

$T = \sum_{i=1}^{n} \frac{1}{x_i}$, a complete sufficient statistic for λ.

Let Z=1/X , the p.d.f of Z

$$h(z) = f\left(x = z^{-1}\right)\left|\frac{dx}{dz}\right|$$

$$h(z) = \lambda e^{-\lambda z}, z > 0$$

Therefore Z follows exp (λ) and hence

$$T = \sum_{i=1}^{n} Z_i \sim \Gamma(n, \lambda)$$

with probability density function л (t), where

$$\pi(t) = \frac{\lambda^n}{\Gamma n} t^{n-1} e^{-\lambda t}, t > 0, \lambda > 0, n > 0$$

Now,

$$E(\frac{1}{T}) = \int_0^\infty \frac{1}{t} \frac{\lambda^n}{\Gamma n} t^{n-1} e^{-\lambda t} dt$$

$$E(\frac{1}{T}) = \frac{\lambda}{n-1}$$

Then $\left[\frac{n-1}{T}\right]$ is unbiased estimator for λ, and T is complete sufficient statistic for λ, By Lehmann-Schaffer theorem, the UMVUE of λ denoted by $\hat{\lambda}_{umvue}$ is given by

$$\widehat{\lambda}_{umve} = \frac{n-1}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n-1}{T}$$

Similarly $\widehat{\alpha}_{umve} = \frac{n-1}{\sum_{i=1}^{n} \frac{1}{y_i}} = \frac{n-1}{T}$

3.3 Minimum Mean square error Estimator (MINMSE)

Minimum mean square error estimator can be found by assuming that:

$\widehat{\lambda}_{MinMSE} = \frac{c}{T}$, where c is unknown value to be found.

MSE ($\widehat{\lambda}_{MinMSE}$)=E($\widehat{\lambda}^2 - 2\lambda\widehat{\lambda} + \lambda^2$)

$$= c^2 E\left[\frac{1}{T}\right]^2 - 2c\lambda E\left[\frac{1}{T}\right] + \lambda^2$$

To minimize partial derivative with respect to (c) and then equating it to zero

$$\frac{\partial}{\partial c} MSE(\hat{\lambda}_{MinMSE}) = 2cE\left(\frac{1}{T}\right)^2 - 2\lambda E\left(\frac{1}{T}\right)$$

$$c = \frac{\lambda E\left(\frac{1}{T}\right)}{E\left(\frac{1}{T}\right)^2} ....(*)$$

$$E\left(\frac{1}{T}\right)^2 = \int_0^\infty \frac{1}{t^2}\frac{\lambda^n}{\Gamma n} t^{n-1} e^{-\lambda t} dt$$

$$= \frac{\lambda^2}{(n-1)(n-2)} \int_0^\infty \frac{\lambda^{n-2}}{\Gamma n-2} t^{n-3} e^{-\lambda t} dt$$

$$E\left(\frac{1}{T}\right)^2 = \frac{\lambda^2}{(n-1)(n-2)}$$

$$E\left(\frac{1}{T}\right) = \frac{\lambda}{(n-1)} \Rightarrow E\left[\frac{n-1}{T}\right] = \lambda$$

Substituting in (*) we get c=n-2

Hence MINMSE for λ is

$$\hat{\lambda}_{MinMSE} = \frac{n-2}{T}$$

Similarly,

$$\hat{\alpha}_{MinMSE} = \frac{n-2}{T}$$

4. Calculation of MSE For three Estimators:

MSE ($\hat{\lambda}$) =E ($\hat{\lambda} - \lambda$)^2

MSE ($\hat{\lambda}_{MLE}$) =E $\left[\frac{n}{T} - \lambda\right]^2$

MSE ($\hat{\lambda}_{MLE}$) = $\frac{\lambda^2 (n+2)}{(n-1)(n-2)}$

$$MSE(\widehat{\lambda}_{UMVE}) = E\left[\frac{n-1}{T}-\lambda\right]^2$$

$$=(n-1)^2 v\left(\frac{1}{T}\right)+\left[E\left(\frac{n-1}{T}\right)-\lambda\right]^2$$

$$v\left(\frac{1}{T}\right)=E\left(\frac{1}{T}\right)^2-\left[E\left(\frac{1}{T}\right)\right]^2$$

$$=\frac{\lambda^2}{(n-1)^2(n-2)}$$

Therefore $MSE(\widehat{\lambda}_{UMVE}) = \dfrac{\lambda^2}{(n-2)}$

Now $MSE(\widehat{\lambda}_{MinMSE}) = E\left[\dfrac{n-2}{T}-\lambda\right]^2$

Therefore $MSE(\widehat{\lambda}_{MinMSE}) = \dfrac{\lambda^2}{(n-1)}$

$MSE(\widehat{\lambda}_{MinMSE}) \le MSE(\widehat{\lambda}_{UMVE}) \le MSE(\widehat{\lambda}_{MLE})$ similarly,

$\widehat{\alpha}_{MinMSE}, \widehat{\alpha}_{UMVE}, \widehat{\alpha}_{MLE}$

5. Reliability of R_4, MLE, UMVUE, MINMSE Estimator, k=0.2 .

α	R(1)	R(2)	R(3)	R(4)	R_4
0.5	0.3967	0.2617	0.1525	0.1176	0.9285
0.6	0.3958	0.2537	0.1304	0.1060	0.8859
0.7	0.3937	0.2253	0.1238	0.0999	0.8427
0.8	0.3824	0.2039	0.1158	0.0987	0.8008
0.9	0.3741	0.1935	0.0935	0.0932	0.7543
1.0	0.3581	0.1780	0.0874	0.0826	0.7061
1.1	0.3536	0.1732	0.0842	0.0587	0.6697
1.2	0.3406	0.1689	0.0698	0.0532	0.6325

Table:1.Marginal and system reliability R_4, λ=0.3

α	R(1)	R(2)	R(3)	R(4)	R_{mle}
0.5	0.3821	0.2411	0.1389	0.1014	0.8635
0.6	0.3798	0.2218	0.1232	0.0923	0.8171
0.7	0.3542	0.2108	0.1112	0.0911	0.7673
0.8	0.3423	0.1909	0.1032	0.0879	0.7243
0.9	0.3369	0.1763	0.0937	0.0732	0.6801
1.0	0.3353	0.1732	0.0832	0.0573	0.6460
1.1	0.3279	0.1689	0.0532	0.0428	0.5928
1.2	0.3128	0.1532	0.0431	0.0327	0.5418

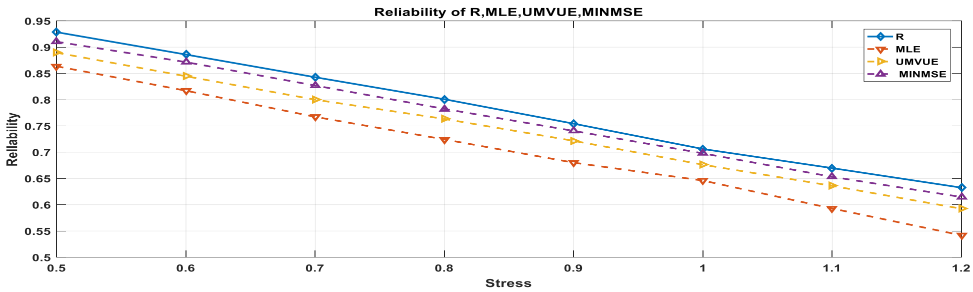
Table:2 Marginal and system reliability R_{mle}

α	R(1)	R(2)	R(3)	R(4)	R_{umvue}
0.5	0.3920	0.2513	0.1446	0.1015	0.8894
0.6	0.3812	0.2313	0.1354	0.0967	0.8446
0.7	0.3769	0.2219	0.1059	0.0953	0.8000
0.8	0.3678	0.2013	0.1053	0.0889	0.7633
0.9	0.3523	0.1889	0.0950	0.0854	0.7216
1.0	0.3426	0.1788	0.0759	0.0789	0.6762
1.1	0.3343	0.1667	0.0456	0.0693	0.6359
1.2	0.3241	0.1653	0.0438	0.0592	0.5924

Table:3.Marginal and system reliability R_{umve}

α	R(1)	R(2)	R(3)	R(4)	R_{minnse}
0.5	0.3923	0.2612	0.1549	0.1019	0.9103
0.6	0.3921	0.2515	0.1265	0.1014	0.8715
0.7	0.3918	0.2289	0.1264	0.0998	0.8269
0.8	0.3712	0.2059	0.1064	0.0989	0.7824
0.9	0.3628	0.1895	0.0981	0.0901	0.7405
1.0	0.3543	0.1785	0.0791	0.0859	0.6978
1.1	0.3452	0.1778	0.0538	0.0763	0.6531
1.2	0.3349	0.1667	0.0448	0.0683	0.6147

Table:4 Marginal and system reliability R_{minmse}



6. Conclusion

The reliability of n-cascade stress-strength model has been obtained by using inverse exponential distribution. Simulation is performed to compare reliability performance of MLE, UMVUE and MinMse estimators. MSE is used to make a comparison between three estimators for each reliability function. From the numerical calculations Minimum mean square error is the best estimator among the MLE and UMVUE.

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