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Study of Parallel Vector Field son Light like Hypersurfaces of Lorentzian Manifolds

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Abstract –In this paper we study parallel vector fields lying on lightlike hypersurfaces$(M, \bar{g})$ of the Lorentzian manifold $(\bar{M}, g)$. We have obtained some results dealing with parallel vector fieldson lightlike hypersurfaces$(M, \bar{g})$ of the Lorentzian manifold $(\bar{M}, g)$. We have discussed totally umbilical, totally umbilical screen homothetic, screen conformaland totally geodesic lightlike hypersurfaces of Lorentzian manifolds admitting parallel vector fields. Furthermore, Ricci soliton lightlike hypersurfaces admitting parallel vector fields are studied and some characterizations for this frame of hypersurfaces are obtained.

Keywords: Parallel vectorfield;Lightlikehypersurface;Lorentzianmanifold;Riccisoliton.

1. Introduction

In 1943, K. Yano [29] proved that there exists a smooth vector field $\boldsymbol{v}$, so called concurrent on a Riemannian manifold $(\bar{M}, g)$ which satisfies the following condition

$$D_F v = F,$$

for every vector field F tangent to$\bar{M}$. Where D is the Levi-Civita connection with respect to Riemannian metric g on$\bar{M}$.The Riemannian and semi-Riemannian manifolds equipped with concurrent vector fields have found its applications in every field of differential geometry and have been intensely studied by various authors ([6], [9]).

Beside these facts, the notion of a Ricci soliton [17] was initially witnessed by Hamilton's Ricci flow and then the Ricci solitons drew attention after G. Perelman [23]. The same author [17] applied Ricci solitons to solve the problem of Poincare conjecture. The Riemannian manifold $(\bar{M}, g)$ equipped with a symmetric Riemannian metric tensor g is called a Ricci soliton [9] if there exists a smooth vector field v tangent to M satisfying the following equation

(1) $$\frac{1}{2} L_v g(F, J) + Ric(F, J) = \lambda g(F, J).$$

In the preceding twenty years, the assumption of geometric flow is the most vital mathematical invention to describe the geometrical configurations in the field of Riemannian geometry. A detailed fragment of description on which the metric develops, diffeomorphisms has substantial influence in the investigation of singularities of the flows as they display up as promising singularity models. These models are commonly authorized soliton solutions.

On the other hand, the yamabe flow was presented by Hamilton [17] at the same time at the emergence of Ricci flow. The Ricci soliton and Yamabe soliton are extraordinary solutions of the Ricci flow and Yamabe flow respectively. For dimension $n = 2$, the Yamabe soliton and the Ricci soliton are alike. For dimension, $n > 2$, the Yamabe soliton and the Ricci soliton are not same and in the enlargement of current existences, the imaginary conception of geometric flows are developed. For instance the Ricci flow and the Yamabe flow has been the crucial matter of interest in the field for numerous geometers.

At the beginning of 2019, Guler and Crasmareanu [16] presented the investigation of alternative geometric flow namely Ricci-Yamabe map. This map is nothing but a scalar combination of the Ricci flow and the Yamabe flow. This is moreover termed to be as (α, β) type Ricci-Yamabe flow. This category of flow is a development for the metrics on the Riemannian manifolds [15] and is well defined by

(2) $$\frac{\delta}{\delta t} g(t) = \beta r(t) g(t) - 2\alpha S(t), \; g_0 = g(0).$$

Where S denotes the Ricci tensor, r represents the scalar curvature, and $\alpha, \beta, \gamma \in \boldsymbol{R}$. One can consider the Ricci-Yamabe flow as singular Riemannian or semi-Riemannian or Riemannian flow because of the representation of symbols α and β as involved scalars. This category of different selections can be respected in some physical or mathematical models such as relativistic theories.

Therefore typically the Ricci-Yamabe solitons are arising as the restriction of the soliton of the Ricci-Yamabe flow. One more solid encouragement that started the exploration of the Ricci-Yamabe solitons is that besides the fact that the Ricci solitons and the Yamabe solitons are equal in $2-$dimensional spaces. In higher dimension they are mainly different. The Ricci-Yamabe soliton on a differentiable manifold $(\bar{M}, g)$ [9] is a data $(g, F, \lambda, \alpha, \beta)$ accomplishing

(3) $$\mathcal{L}_F g(F,J) = -2\alpha S(F,J) - (2\lambda - \beta r) g(F,J),$$

where$S$specifies the Ricci tensor, £ represents the Lie-derivative, $r$represents the scalar curvature and $\lambda,\ \alpha,\ \beta\ \in \boldsymbol{R}$. If $f$symbolizes a smooth function and F is the gradient of f, then the above mentioned notion is termed to be as the gradient Ricci-Yamabe soliton [9] and the above equation (3) turn out to be

(4) $$\bar{D}^2 f = -\alpha S(F,J) - \left(\lambda - \frac{1}{2}\beta r\right) g(F,J).$$

Where $\bar{D}^2 f$ denoted the hessian of f.

The gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) is expanding for $\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$.If$\lambda,\ \beta$and $\alpha$are smooth functions on $\bar{M}$ it may be noted that the gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) is called an almost gradient Ricci-Yamabe soliton (or almost Ricci-Yamabe soliton).

For $\beta = 0,\ \alpha = 1$, then gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) turns into gradient Ricci soliton (or Ricci soliton) [16]. Similarly, for $\beta = 1, \alpha = 0$, then gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) turns into gradient Yamabe soliton (or Yamabe soliton) [17]. On the other hand gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) shrinks to gradient Einstein soliton (or Einstein soliton) [7] for $\beta = -1$ and $\alpha = 1$.

Duggal-Bejancu [12] introduced the concept of lightlike geometry of semi-Riemannian manifolds and is wholly different from Riemannian and semi-Riemannian manifolds. To overcome this difficulty ascended due to degenerate metric authors [11] obtained transversal bundle for such hypersurfaces. After ([11], [12]) researchers across the globe studied lightlike hypersurface of manifolds by succeeding Duggal-Bejancu methodology. For degenerate hypersurfaces of manifolds we refer ([11], [12], [13], [14], [22]) any many more references therein.

The main purpose of this paper is to examine the behavior of parallel vector fields or parallel fields on lightlike hypersurfaces and Ricci solitons lightlike hypersurfaces of a Lorentzian manifolds. However there are few problems to deal with while investigating parallel vector fields and Ricci solitons for these kinds of hypersurfaces.

The principal difficult is the existence of degenerate induced metric and hence not applicable for a lightlike hypersurface. Some noteworthy differential operators such as the divergence, Laplacian operators and gradient with respect to the degenerate metric cannot be explained. To overcome of this problem with the help of a rigging vector field we study the associated metric defined on such hypersurfaces. The subsequent main difficulty is that the Ricci tensor of any degenerate hypersurface is not symmetric and in this situation the equation of Ricci soliton loses its geometric and physical senses. To overcome this situation, we explored this Ricci soliton equation on degenerate hypersurfaces of Lorentzian manifolds whose Ricci tensor is symmetric.

2. **Preliminaries**

In 1998, Petersen [24] showed there exists a vector field $v$so called parallel vector fieldon a Riemannian manifold Mif for any F tangent to M, one has

(5) $$D_F v = 0,$$

where$D$ is the Levi-Civita connection and represents covariant derivative of parallel vector field v in the direction of F.On the other hand the notion of concurrent vector fields can naturally be extended to semi-Riemannian manifolds. In that situation Chen [8] proved that a Lorentzian manifold is a generalized Robertson–Walker spacetime if and only if it admits a timelike concircular vector field.

Concircular vector fields are also important in the study of concircular mappings that is conformal mappings are preserving geodesic circles [27]. In physics, particularly in general relativity concircular vector fields have interesting applications.

Now let us recall some of the elementary and important terminologies about the geometry of lightlike (degenerate) hypersurfaces of semi-Riemannian manifolds. Assume that $(M, \bar{g}, S(TM))$ to be a null hypersurface of $(\bar{M}, g)$. Then over this on $(M, \overline{g}, S(TM))$ there exists $tr(TM)$ a rank 1 unique vector bundle in such a way that for any Z of $TM^{\perp}$ on $Y \subset M$. Radical distribution at a point may be defined as

(6) $$Rad\ T_P M = \{\xi \in\ T_P M: \bar{g}(\xi, F) = 0,\ \forall\ F \in TM\}.$$

Then there exists X a unique section of $tr(TM)$ on the coordinate neighborhood Yknown as null transversal vector field of hypersurface $(M,\ \overline{g},\ S(TM))$. Such that

(7) $$\bar{g}(Z,\xi) = 1, \qquad \bar{g}(X,X) = \bar{g}(X,F)) = 0,$$

$\forall\ F \in \Gamma(S(TM_{|M}))$. Then tangent bundle $T\bar{M}$ is decomposed as

$$T\bar{M}_{|M} = S(TM) \oplus \left(TM \perp \oplus\ tr(TM)\right),$$

(8) $$T\bar{M}_{|M} = TM \oplus tr(TM).$$

Here $tr(TM)$ is known to be as lightlike transversal bundle of hypersurface with respect to $S(TM)$ and $tr(TM)$ is complementary but not orthogonal vector bundletoTMin $\bar{M}_{|M}$[9]. By considering equation (8) for all $F, J \in \Gamma(S(TM_{|M})$ the local Gauss and Weingarten formulas are given by

(9) $$\bar{D}_F J = D_F J + B(F, J),$$

(10) $$\bar{D}_F X = -A_X F + \tau(F)X,$$

(11) $$D_F PJ = D^*{}_F PJ + C(F, PJ)\xi,$$

(12) $$D_F Z = A^*{}_Z F + \tau(F)\, Z$$

Here $\bar{D}, (D, D^*)$ represent Livi-Civita connection of $(\bar{M}, g)$ and linear connections on $(TM,\ S(TM))$ respectively. (B, C)and(A_X, $A^*{}_X$) represent local fundamental forms and shape operators on TM and $\Gamma(S(TM)$ respectively. Also $\tau(F) = g(D_F^t X, \xi)$ and P represent 1-form and projection morphisms of Γ(TM) on S(TM) respectively. By using the fact that $B(F, J) = g(\bar{D}_F J\,, \xi)$, we know that the local second fundamental form B is independent of choice of S(TM) and hence satisfies

(13) $$B(F, Z) = 0, \quad \forall\, F \in \Gamma\,(TM).$$

Unfortunately D on TM is not a metric connection and hence satisfies

(14) $$(D_F \bar{g})(J, L) = B(F, J)\theta\,(L) + B(F, L)\theta\,(J) \quad \vee F, J, L \in \Gamma\,(TM).$$

Here θ represents 1- form defined by

$$\theta(F) = g(F, X),$$

for all $F \in \Gamma\,(TM)$.The Lie derivative of g with respect to the its Levi-Civita connection $\bar{D}$ is defined by

(15) $$(L_F g)(J, L) = g(D_F J, L) + g(D_F L, J)$$

$\forall\ \ F,\ J,\ L \in \Gamma\,(TM)$.For any lightlike hypersurface $(M, \bar{g}, S(TM))$ of Lorentzian manifold $(\bar{M}, g)$, we have from equations (13) and (14)

(16) $$(L_F \bar{g})(J, L) = B(F, J)\theta\,(L) + B(F, L)\theta\,(J) + \bar{g}(D_F J, L) + \bar{g}(D_F L, J),$$

$\forall\ F,\ J,\ \ L \in \Gamma\,(TM)$or equivalently above equation may be written as

(17) $$(L_F \bar{g})(J, L) = (D_F \bar{g})(J, L) + \bar{g}(D_F J, L) + \bar{g}(D_F L, J)\ \vee F, J, L \in \Gamma\,(TM).$$

However D^*on $S(TM)$ is a metric connection. The above shape operators are related to their local second fundamental forms [11] as

(18) $$B(F, J) = \bar{g}(A^*{}_X\, F, J), \quad \bar{g}(A^*{}_X F, X) = 0,$$

(19) $$C(F, PJ) = \bar{g}(A_X F, PJ), \quad \bar{g}(A_X F, X) = 0.$$

From equation (16) $A^*{}_Z\, F = 0$. With respect to connections $\bar{D}$ and D the Riemannian curvature tensors of $(\bar{M}, g)$ and $(M, \bar{g}$, $S(TM))$ are represented by $\bar{R}$ and R respectively as given by

(20) $$g(\bar{R}(F, J)\xi, PO) = \bar{g}(R(F, J)L, PO) + B(F, L)C(J, PO) - B(J, L)C(F, PO),$$

(21) $$g(\bar{R}(F, J)L, \xi) = g^{-}(R(F, J)\xi, PO) = (D_F\, B)(J, L) - (D_J\, B)(F, L) + B(J, L)\ \tau(F) - B(F, L)\tau\,(J),$$

(22) $$g(\bar{R}(F, J)L, X) = \bar{g}(R(F, J)\xi, X),$$
$$= \bar{g}(D_F(A_X J) - D_J(A_X F) - \bar{g}(A_X(F, J), L) + \bar{g}(A_X F, L)\tau\,(J)$$
$$-\bar{g}(A_X J, L)\tau(F) + \bar{g}\left(A^*{}_\xi\, F, A_X J\right) - \bar{g}\left(A^*{}_\xi\, J, A_X F\right) - 2d\tau(F, J)\theta(L),$$

(23) $$(D_F g)(J, L) = B(F, J)\theta\,(L) + B(F, L)\theta\,(J).$$

$\forall\, F, J, L \in \Gamma\,(TM)$.

Definition: Let $(M, \bar{g})$ be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, g)$. A point p in M is umbilic, if $B_P(F_P, J_P) = k\bar{g}_p(F_P, J_P)$, for $F_p,\ J_p\ \in M$, where B represents local second fundamental form. $M$is totally umbilic if any point of hypersurface is umbilic. It is not difficult to check that hypersurface M is totally umbilic if locally on each point in M there exists a smooth function φ, such that$B(F, J) = \varphi\bar{g}(F, J)$, for any F and J tangent to M. In case$\varphi = 0$ one says M is totally geodesic.

Definition:A null hypersurface $(M, \bar{g})$ of a semi-Riemannian manifold is screen locally conformal, if the shape operators $A_X$and A^*_ξ of $M$and $S(T\ M)$ respectively are related by $A_X = \phi A^*_\xi$. If $\phi$is non-zero constant then $M$is screen homothetic.

Definition: Hypersurface $(M, \bar{g}, S(TM))$ of $(\bar{M}, g)$is of genus zero [10] with screen $S(TM)$ if

- M,admits a unique screen distribution $S(TM)$ that makes a unique N and null hypersurface Madmits an induced symmetric Ricci tensor.
- The Ricci tensor $R^{(0,2)}$ be symmetric on $(M, \bar{g})$.

Definition: The smooth manifoldis said to be as an Einstein degenerate hypersurface [14] if for any vector fields $F, J \in \Gamma(TM)$, the resulting relation is satisfied

(24)
$$R^{(0,2)}(F,J) = \gamma\, \bar{g}(F,J),$$

where$\gamma$is constant.

3. Parallel Vector Fields

For any degenerate hypersurface $(M, \bar{g}, S(TM))$, some noteworthy differential operators such as the gradient could be defined by the assistance of its associated metric and a rigging vector field. Therefore, we shall firstly remember some basic truths linked to rigging vector fields and their some elementary properties before take in parallel vector fields on $(M, \bar{g}, S(TM))$.

Let $(M, \bar{g}, S(TM))$ be degenerate hypersurface of a Lorentzian manifold $(\bar{M}, g)$ and ξ be a vector field well-defined in an open set containing M. Let us assume that $\xi_p \notin Tp\ M$ for any $p \in M$. If there exists a 1-form η such that $\eta(F) = g(F, \xi)$ for any vector field $F \in \Gamma(TM)$, then ξ is a rigging vector field for M. Now let $X \in tr(TM)$ be a rigging vector field for M and ηbe a 1-form defined by

(25)
$$\eta(F) = g(F, X),$$

for any $F \in \Gamma(TM)$. In this situation one can express a (0,2) type metric tensor $g$as follows:

(26)
$$g(F,J) = \bar{g}(F,J) + \eta(F)\eta(J),$$

for any $F, J \in TM$. It may be noted that the associated metric $\bar{g}$is non-degenerate. Therefore from equations (6), (7) and (26), we acquire

(27)
$$g(\xi,\xi) = 1, \ \ g(\xi, F) = \eta(F),$$

and

(28)
$$\bar{g}(F,J) = g(F,J), \ \ \eta(F) = 0,$$

$\forall \ F, J \in \Gamma(S(TM))$. Now let v be the parallel vector field on $\Gamma(T\bar{M})$ then we can describe parallel vector field $v$as the tangential and transversal constituents by

(29)
$$v = V^T + V^X,$$

where$V^T \in \Gamma(TM)$ and $V^X \in tr(TM)$. From (26) and (29), we have

$$g(V^T, \xi) = \bar{g}(V^T, \xi) + \eta(V^T)\eta(\xi)),$$
$$g(V^T, \xi) = \eta(V^T),$$

(30)
$$g(V^T, \xi) = g(V^T, X).$$

for any $F \in \Gamma(TM)$, we write

(31)
$$Fg(V^T + \xi) = Fg(V^T + X),$$

(32)
$$= g(D_F v + X) + g(v, D_F X),$$

(33)
$$Fg(V^T + \xi) = \tau(F)\eta(v) - g(v, A_X F).$$

Now let us assume v lies in tangent bundle i,e. $v = V^T$. So for any $v^S \in \Gamma(S(TM))$ and $\eta(v) = a$, we can write

(34)
$$v = v^S + a\xi.$$

4. Results and Discussion

Lemma 1-Let v be the parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of a Lorentzian manifold $(\bar{M}, g)$, then

(35)
$$\tau(v) = \frac{1}{a} g(v^S, A_X v).$$

Proof- From equations (30), (33) and (34), we acquire

$$(v^S + a\xi)g(v^S + a\xi, \xi) = g(v^S + a\xi, v)g(v^S + a\xi, X)$$
$$-g(v^S + a\xi, X) - g(v^S + a\xi, A_X(v^S + a\xi)) +$$
$$\tau(v^S + a\xi)g(v^S + a\xi, X),$$

$$(v^S + a\xi)[g(v^S, \xi) + a(v^S + a\xi)g(\xi, \xi)] = \tau(v^S + a\xi)[\eta(V^T) + a\eta(\xi)].$$

By straightforward computation of the above equation, we acquire

$$a\tau(v) = g(v^S, A_X v) + ag(v^S, A_X \xi).$$

From the above equation, we obtain (35).

Lemma 2- Let v be the parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$. Then for any $F \in \Gamma(TM)$

(36) $$D_F v = 0,$$

$$B(F, v) = 0.$$

Proof- From equations (5) and (9) we obtain

$$\bar{D}_F v = D_F v + B(F, v),$$

by using equation (9) in the above equation, we obtain

(37) $$D_F v + B(F, v)X = 0.$$

Comparing tangential and transversal parts of the above equation (37), we obtain equations (36) and (37).

Lemma 3- Tangent to the structural vector field let $(M, \bar{g}, S(TM))$be screen conformal null hypersurface of Lorentzian manifold$(\bar{M}, g)$. Let v be the parallel vector field on $\Gamma(TM)$ then

$$\tau(v) = 0,$$

for any$F \in \Gamma(TM)$.

Proof-.From equations (9), (32) and lemma-1, we obtain

(38) $$\tau(v) = \frac{1}{a}C(v, V^S).$$

As assumed$(M, \bar{g}, S(TM))$ is screen conformal null hypersurface then equation (38), becomes

(39) $$\tau(v) = \frac{1}{a}\frac{1}{\varphi}B(v, V^S).$$

Using equation (36) in equation (39), we obtain our required result.

Lemma 4- Let v be the parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface of Lorentzian manifold then

(40) $$D^*_\xi v = 0 \quad and C(\xi, v) = 0.$$

Proof- As second fundamental form B vanishes on $Rad(TM)$, then using the fact

$$D_F J = D^*_F J + C(F, J)\xi,$$

we obtain

$$\bar{D}_\xi v = D_\xi v,$$

$$D_\xi v = D^*_\xi v + C(\xi, v)\xi.$$

By making use of equations (5) in the forgoing equation, we acquire

(41) $$D^*_\xi v + C(\xi, v) = 0,$$

by comparing like terms of the above equation (41), we obtain (40).

Theorem 1- Let $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface and let v be the parallel vector field lying on$\Gamma(TM)$. Then $\tau(\xi) = 0$.

Proof- From lemma 3 and the Guass Weingarten formulas, we obtain

(42) $$D_v \xi = -A^*_\xi v$$

(43) $$\bar{D}_\xi X = -A_X v$$

Since $v$is a parallel vector field, therefore using lemma 4, we achieve

(44) $$D_\xi v = 0.$$

From equations (42) and (44), we acquire

(45) $$D_\xi v - D_v \xi = -A_\xi^* v,$$

(46) $$[\xi, v] = -A_\xi^* v.$$

Letting$v = v^S + a\xi$, then above equation turn into

(47) $$[\xi, v^S + a\xi] = -A_\xi^*(v^S + a\xi),$$

(48) $$[\xi, v^S] = -A_\xi^* v^S - aA_\xi^* \xi ,$$

(49) $$D_\xi v^S - D_{v^S} \xi = -A_\xi^* v^S - aA_\xi^* \xi.$$

Using Guass and Weingarten formulas in above equation, we attain

(50) $$D_\xi^* v^S + C(\xi, v^S)\xi - A_\xi^* v^S + \tau(v^S)\xi = -A_\xi^* v^S - aA_\xi^* \xi.$$

Since$A_\xi^* \in S(TM)$, therefore from equation (50), we acquire

(51) $$\tau(v^S)\xi + C(\xi, v^S)\xi = 0.$$

Using lemma 4 in the forgoing equation, we attain

(52) $$\tau(v^S) = 0.$$

Using $\tau(v) = 0$ in the above equation, we obtain

(53) $$\tau(v) = \tau(v^S + a\xi) = 0,$$

$$a\tau(\xi) = 0.$$

Hence from the above equation we conclude that either $a = 0$ or$\tau(\xi) = 0$. But $a = \eta(v) \neq 0$, hence$\tau(\xi) = 0$.

Theorem 2-Let v be parallel vector field with respect to Levi-Civita connection$\bar{D}$. Then correspondingly v is parallel vector field with respect to induced Riemannian connection D^1.

Proof- From equation (26), for any $F \in \Gamma(TM)$, we have

$$Fg(v, v) = F\bar{g}(v, v) + F(g(v, N)^2)$$

(54) $$Fg(v, v) = 2g(D_F v, v) + 2\eta(v)[-g(v, A_X F) + \tau(F)\eta(v)].$$

Using the fact, $Fg(v, v) = 2\bar{g}(D_F^1 v, v)$, in the above equation we acquire

(55) $$Fg(v, v) = 2\bar{g}(D_F^1 v, v) + 2\eta(D_F^1 v)\eta(v).$$

From equations (54) and (55), we acquire

(56) $$D_F^1 v + \eta(D_F^1 v)X = [\tau(F)\eta(v) - g(v, A_X F)]X.$$

Comparing the corresponding parts on both sides we obtain

(57) $$D_F^1 v = 0,$$

and

(58) $$\eta(D_F^1 v) = \tau(F)\eta(v) - g(v, A_X F).$$

Equation (57), shows that v is also a parallel vector field with respect toD^1. From equation (58), we may state the following corollaries

Corollary 1-Let v be a parallel vector field with respect toD^1. Then v is also parallel with respect to D if and only if the following condition

$$\eta(D_F^1 v) = \tau(F)\eta(v) - g(v, A_X F),$$

is satisfied for all $X \in \Gamma(TM)$.

Corollary 2- Let v be the parallel vector field with respect toD^1 lying on $\Gamma(TM)$ and $(M,\ \bar{g},\ S(TM))$ be screen conformal null hypersurface with respect to$D$. Then v is also parallel with respect to D if and only if the following condition

$$\eta(D_V^1 v) = -g(v, A_X F).$$

Theorem 3- Let $(M,\bar{g})$ be totally umbilical screen-homothetic null hypersurface of $\bar{M}(c)$ admitting parallel vector field v on$\Gamma(T\bar{M})$, then $(M,\bar{g})$ is semi-Euclidean.

Proof- By using given assumption and theorem-1 that$\tau(\xi)=0$, we acquire

$$D_\xi \xi = A^*_\xi \xi,$$

but

$$B(\xi, J) = \bar{g}(A^*_\xi \xi, J).$$

From the above equation, it follows that$A^*_\xi \xi = 0$and $D_\xi \xi = 0$.

The Riemannian curvature tensor R equipped with parallel vector field v is given by

$$R(\xi,v)\xi = D_\xi D_v \xi - D_v D_\xi \xi - D_{[\xi,v]}\xi.$$

By using equation (46) in the above equation, we get

$$R(\xi,v)\xi = D_{A^*_\xi v}\xi.$$

By using the given fact that hypersurface $(M,\bar{g})$ is totally umbilical, we acquire

$$R(\xi,v)\xi = D_{\lambda v}\xi,$$

$$R(\xi,v)\xi = \lambda D_v \xi.$$

Using equation (42) in the above equation, we obtain

$$R(\xi,v)\xi = -\lambda(A^*_\xi v).$$

From equation (20), we find

$$\bar{g}(R(\xi,v)\xi, X) = \lambda \bar{g}(D_v\xi, X),$$

and

$$\bar{g}(R(\xi,v)\xi, X) = 0.$$

From the above equation, it is clear that $(M,\bar{g})$is semi-Euclidean.

5. Ricci Solitons on Null Hypersurfaces

Let $(M,\bar{g},S(TM))$ be the null hypersurface of $(\bar{M},g)$ and v be the parallel vector field on tangent bundle of $(\bar{M},g)$. Then v can be written as

(59) $$v = V^T + fX,$$

whereV^T and fX are tangential and transversal components of v. Also $V^T \in \Gamma(TM)$ and $f = g(v,\xi)$.

Theorem 4. Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$, then

(60) $$D_F v^T = f A_X F,$$

(61) $$B(F, v^T) = f\tau(F) - Ff.$$

Proof. From equation (59), we acquire

(62) $$\tau(v) = \tau(V^T + fX).$$

As assumed v is a parallel vector field therefore

(63) $$\bar{D}_F v = 0,$$

(64) $$\bar{D}_F v^T + \bar{D}_F(fX) = 0.$$

Using Guass and Weingarten formulas in the forgoing equation, we achieve

(65) $$D_F v^T + B(F,v^T)X + F(f)X - fA_X F - f\tau(F)X = 0.$$

From equations (63) and (65), we find equations (60) and (61).

Theorem 5. Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be a totally geodesic null hypersurface of $(\bar{M},g)$. Then either the function f is constant or the Lie bracket is trivial.

Proof. From equation (61), if null (lightlike) hypersurface $(M,\bar{g},S(TM))$ is totally geodesic then we acquire

(66) $$f\tau(F)=Ff,$$

Similarly

(67) $$f\tau(J)=Jf.$$

Subtracting equations (67) and (68), we obtain

(68) $$[F,J](f)=f[J(\tau(F))-F(\tau(J))]$$

for any $X,Y\in\Gamma(TM)$. Using corollary 1 and proposition 2 of [18] in equation (68), we achieve

$$[F,J](f)=0.$$

The above equation signifies, either the function f is constant or the lie bracket is trivial.

Lemma 6. Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$, thenfor any $F,J\in\Gamma(TM)$ we have

(69) $$(L_{v^T})\bar{g}(F,J)=B(v^T,F)\theta(J)+B(v^T,J)\theta(F)+f\bar{g}(A_XF,J)+f\bar{g}(A_XJ,F),$$

or equivalently

(70) $$(L_{v^T})\bar{g}(F,J)=(D_{v^T}\bar{g})(F,J)+f[\bar{g}(A_XF,J)+\bar{g}(A_XJ,F)].$$

Proof. From equations (16) and (17), we obtain

(71) $$(L_{v^T}\bar{g})(F,J)=B(v^T,F)\theta(J)+B(v^T,J)\theta(F)-\bar{g}(D_Fv^T,J)-\bar{g}(D_Jv^T,F),$$

or equivalently

(72) $$(L_{v^T}\bar{g})(F,J)=(D_{v^T}\bar{g})(F,J)+\bar{g}(D_Fv^T,J)+\bar{g}(D_Jv^T,F).$$

By making use of equations (59) and (60) in equations (71) and (72), we achieve our results.

If the parallel vector field v lies on $\Gamma(T\bar{M})$, that is if $(f=0)\Rightarrow V^T=v$, then we may state

Lemma 6. Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$, thenfor any $X,Y\in\Gamma(TM)$, we have

(73) $$(L_{v^T}\bar{g})(F,J)=(D_{v^T}\bar{g})(F,J).$$

Proposition 1. Let v^T be a potential vector field and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$. Thenfor any $X,Y\in\Gamma(TM)$, we have

(74) $$2R^{(0,2)}(F,J)=2\lambda\bar{g}(F,J)-f[\bar{g}(A_XF,J)+\bar{g}(A_XJ,F)]-(D_{v^T}\bar{g})(F,J),$$

or equivalently

(75) $$R^{(0,2)}(F,J)=\lambda\bar{g}(F,J)-\frac{f}{2}[\bar{g}(A_XF,J)+\bar{g}(A_XY,F)]-\frac{1}{2}(D_{v^T}\bar{g})(F,J).$$

Proof. We know that if $(M,\bar{g},S(TM))$ is Ricci soliton lightlike hypersurface then it satisfies

(76) $$(L_{v^T}\bar{g})(F,J)+2R^{(0,2)}(F,J)=2\lambda\bar{g}(F,J).$$

Using lemma 5, in the above equation we obtain our required results.

Proposition 2. Let v be a potential vector field and $(M,\bar{g},S(TM))$ be a Ricci soliton null (lightlike) hypersurface of a semi-Euclidean space of dimension $(n-1)$. Thenfor any $F\in\Gamma S((TM))$, we have

(77) $$nB(F,F)traceA_X-B(A_XF,F)=\lambda-f\bar{g}(A_XF,F),$$

where F is a unit vector field.

Proof. Let us assume that ($a_1, a_2, a_3 \ldots\ldots\ldots a_n$) be orthonormal basis. Therefore from equation (83) and [18], we obtain

(78) $$R^{(0,2)}(a_i,a_i)=nB(a_i,a_i)traceA_X-B(A_Na_i,a_i).$$

Using equation (75) in equation (78), we obtain

(79) $$nB(a_i,a_i)traceA_X-B(A_Xa_i,a_i)=\lambda\bar{g}(a_i,a_i)-f\bar{g}(A_Xa_i,a_i)-\frac{1}{2}(D_{v^T}\bar{g})(a_i,a_i).$$

Replacing a_i with F, we obtain

(80) $$nB(F,F)traceA_X-B(A_XF,F)=\lambda-f\bar{g}(A_XF,F).$$

Corollary 3. Let v be a parallel vector field and $(M, \bar{g}, S(TM))$ be Ricci soliton null (lightlike) hypersurface of semi-Euclidean space. Thenfor any $F, J \in \Gamma S((TM))$, if $(M, \bar{g}, S(TM))$ is screen conformal and totally umbilical then we have

(81) $$n^2[B(F,F)]^2\varphi - [B(F,F)]^2 = \lambda - 2fB(F,F),$$

where $F\epsilon(S(T\bar{M}))$ is a unit vector field.

Proposition 3.Let v be a parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM)$ be Ricci soliton null (lightlike) hypersurface of a Lorentzian manifold. Thenfor any $F, J \in \Gamma(S(TM))$, we have

(82) $$R^{(0,2)}(F,J) = \lambda\bar{g}(F,J) - \frac{1}{2}[B(v,F)\theta(J) + B(v,J)\theta(F)].$$

Proof. Since $(M, \bar{g}, S(TM))$ is a Ricci soliton and v is a parallel vector field lying on $\Gamma(TM)$ tangent bundle. Therefore equation (76) can be written as

(83) $$(L_v\bar{g})(F,J) + 2R^{(0,2)}(F,J) = 2\lambda\bar{g}(F,J).$$

For any $X, Y \in \Gamma(TM)$ and from lemma 6, we have

(84) $$(L_v\bar{g})(F,J) = (D_v\bar{g})(F,J) + 2[B(F)B(J) - \bar{g}(F,J).$$

Using equations (83) in (84), we obtain

(85) $$R^{(0,2)}(F,J) = \lambda\bar{g}(F,J) - \frac{1}{2}(L_v\bar{g})(F,J).$$

Therefore from equations (14) and (85) the proof is straightforward.

Corollary 4.Let v be a parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM)$ be Ricci soliton null (lightlike) hypersurface of a Lorentzian manifold. Then for any $F, J \in \Gamma(TM)$ if $(M, \bar{g}, S(TM)$ is totally geodesic then $(M, \bar{g}, S(TM)$ is Einstein Ricci soliton degenerate hypersurface.

Proof. Let us assume that $(a_1, a_2, a_3, \dots \dots \dots .. a_n, \xi)$ and $(a_1, a_2, a_3, \dots \dots \dots .. a_n)$ be basis on Γ(TM) and $\Gamma(S(TM))$ respectively. Then

(86) $$R^{(0,2)}(a_i, a_j) = \lambda\bar{g}(a_i, a_j) - \frac{1}{2}[B(v, a_i)\theta(a_j) + B(v, a_j)\theta(a_i)].$$
$$R^{(0,2)}(a_i, a_j) = \lambda\delta_{ij}$$

where $i,\ j = 1, 2, 3, \dots \dots \dots \dots n$ and δ_{ij} represents the Kronecker delta.

Putting $F = J = \xi$ in equation (82), we acquire

(87) $$R^{(0,2)}(\xi,\xi) = \lambda\bar{g}(\xi,\xi) - \frac{1}{2}[B(v,\xi)\theta(a_j) + B(v,\xi)\theta(a_i)].$$

(88) $$R^{(0,2)}(\xi,\xi) = \lambda.$$

Now from equations (86) and (88), we obtain our required result.

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