ISSN: 0974-5823 Vol. 7 No. 09 September, 2022

International Journal of Mechanical Engineering

Univalent Analytic Functions With Negative Coefficients Of Complex Order Defined By Geganbauer Polynomial

Pasunoori Srinivasulu
Department of Mathematics, Dr.B.R.Ambedkar Open University, Hyderabad-500033. Telangana, India.
E.mail: srinivasrgukt1203@gmail.com

V.Srinivas
Department of Mathematics, Dr.B.R.Ambedkar Open University, Hyderabad-500033. Telangana, India.

Abstract:
In this paper we define a new class of functions $M_{\lambda}^{b}(A,B,\upsilon,t)$ where functions in this class satisfy the condition $1+\frac{1}{b}\left\{\frac{z(G_{\upsilon,t}f(z))'}{G_{\upsilon,t}f(z)}-1\right\} \prec (1-\lambda)\frac{1+Aw(z)}{1+Bw(z)}+\lambda$, $(w(z)\in E)$. where $\prec$ denotes subordination, b is any non zero complex number, A and B are the arbitary constants $-1\le B<A\le 1$, , $\lambda(0\le\lambda<1)$, $t\in[-1,1]$ and $\upsilon\ge 0$. Coefficient estimates, growth and distortion theorems for this class of functions are found. Radii of convexity, starlikeness, close-to-convexity and convex linear combinations are obtained for this class also.

Keywords: Analytic, Starlike Convex, Subordination, Distortion.

2010 subject Classification: 30 C45.

1. Introduction:

Let A denote the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk $E=\{z\in\mathbb{C}:|z|<1\}$.

A function f in the class A is said to be in the class $ST(\alpha)$ of starlike functions of order α in E, if it satisfy the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\ (0\le\alpha<1), \quad (z\in E) \qquad (1.2)$$

Note that $ST(0)=ST$ is the class of starlike functions.

Denote by T the subclass of A consisting of functions f of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \quad (a_n\ge 0). \qquad (1.3)$$

This subclass was introduced and extensively studied by Silverman [6].

The class $T(\upsilon),\upsilon\ge 0$ were introduced and investigated by Szynal [10] as the subclass of A consisting of functions of the form

$$f(z)=\int_{-1}^{1}k(z,t)d\mu(t). \qquad (1.4)$$

where

$$k(z,t)=\frac{z}{(1-2tz+z^2)^{\upsilon}} \quad t\in[-1,1],\ (z\in E). \qquad (1.5)$$

And μ is a probality measure on the interval [-1,1]. The collection of such measure on [a,b] is denoted by $P_{[a,b]}$.

The Taylor series expansion of the function in (1.5) gives

$$k(z,t)=z+c_1^{\upsilon}(t)z^2+c_2^{\upsilon}(t)z^3+... \qquad (1.6)$$

And the coefficients for (1.6) were given below:

$$c_0^{\upsilon}(t)=1,\ c_1^{\upsilon}(t)=2\upsilon t,\ c_2^{\upsilon}(t)=2\upsilon(\upsilon+1)t^2-\upsilon,\ c_3^{\upsilon}(t)=\frac{4}{3}\upsilon(\upsilon+1)(\upsilon+2)t^3-2\upsilon(\upsilon+1)t,... \qquad (1.7)$$

Where $c_n^{\upsilon}(t)$ denotes the Gegenbauer polynomial of degree n. Varying the parameter υ in (1.6), we obtain the class of typically real functions studied by [1], [4], [5], [9] and [12].

For $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$, the Hadamard product of f and g is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\quad (z\in E).$$

Let $G_{\upsilon,t}:A\to A$ defined in terms of the convolution by

$$G_{\upsilon,t}f(z)=k(z,t)*f(z)\text{, We have }\quad G_{\upsilon,t}f(z)=z+\sum_{n=2}^{\infty}\omega_{n-1}^{\upsilon}a_nz^n, \qquad (1.8)$$

In this paper we define a new class of functions $M_{\lambda}^{b}(A,B,\upsilon,t)$ where functions in this class satisfy the condition

$$1+\frac{1}{b}\left\{\frac{z(G_{\upsilon,t}f(z))'}{G_{\upsilon,t}f(z)}-1\right\} \prec (1-\lambda)\frac{1+Aw(z)}{1+Bw(z)}+\lambda,(w(z)\in E). \qquad (1.9)$$

where $\prec$ denotes subordination , b is any non zero complex number, A and B are the arbitary constants $-1\le B<A\le 1$, $\lambda(0\le\lambda<1)$, $t\in[-1,1]$ and $\upsilon\ge 0$ .Coefficient estimates growth and distortion theorems , radii of convexity, starlikeness ,close-to-convexity and convex linear combinations are obtained for this class.

2. Coefficient Estimates

Theorem 1. A necessary and sufficient condition for a function $f\in T$ to be in the

class $f\in M_{\lambda}^{b}(A,B,\upsilon,t)$ is

$$\sum_{n=2}^{\infty}\left[(n-1)+\left|b(A-B)(1-\lambda)-B(n-1)\right|\right]\omega_{n-1}^{\upsilon}(t)\left|a_n\right|\le\left|b\right|(A-B)(1-\lambda) \qquad (1.10)$$

Proof. By definition of subordination , we can write (1.9) as

$$1+\frac{1}{b}\left(\frac{z(G_{\upsilon,t}f(z))'}{G_{\upsilon,t}f(z)}-1\right) = (1-\lambda)\frac{1+Aw(z)}{1+Bw(z)}+\lambda,(w(z)\in E).$$

which gives

$$\left(\frac{z(G_{\upsilon,t}f(z))'}{G_{\upsilon,t}f(z)}-1\right) = \left[b(A-B)(1-\lambda)-B\left(\frac{z(G_{\upsilon,t}f(z))'}{G_{\upsilon,t}f(z)}-1\right)\right]w(z) \qquad (1.11)$$

From (1.11) , we obtain

$$\frac{z-\sum_{n=2}^{\infty}n(\omega_{n-1}^{\upsilon}(t)a_nz^n}{z-\sum_{n=2}^{\infty}\omega_{n-1}^{\upsilon}(t)a_nz^n}-1 = \left[b(A-B)(1-\lambda)-B\left(\frac{z-\sum_{n=2}^{\infty}n\omega_{n-1}^{\upsilon}(t)a_nz^n}{z-\sum_{n=2}^{\infty}\omega_{n-1}^{\upsilon}(t)a_nz^n}\right)\right]w(z)$$

which yields

$$\frac{\sum_{n=2}^{\infty}-(n-1)\omega_{n-1}^{\upsilon}(t)a_nz^n}{z-\sum_{n=2}^{\infty}\omega_{n-1}^{\upsilon}(t)a_nz^n} = \left[b(A-B)(1-\lambda)-B\left(\frac{\sum_{n=2}^{\infty}-(n-1)\omega_{n-1}^{\upsilon}(t)a_nz^n}{z-\sum_{n=2}^{\infty}\omega_{n-1}^{\upsilon}(t)a_nz^n}\right)\right]w(z)$$

Since $|w(z)|<1$,

$$\left|\sum_{n=2}^{\infty}-(n-1)\omega_{n-1}^{\upsilon}(t)a_nz^n\right| \le \left|b(A-B)(1-\lambda)z-\sum_{n=2}^{\infty}\left[b(A-B)(1-\lambda)-B(n-1)\right]\omega_{n-1}^{\upsilon}(t)z^n\right|$$

Letting $|z|\to 1$, we have

$$\sum_{n=2}^{\infty}\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)|a_n| \;\le\; |b|(A-B)(1-\lambda).$$

Conversely, let (1.10) be true. From (1.11), we see that $|w(z)|<1$,

$$\left|\frac{z(G_{\upsilon,t}f(z))'-G_{\upsilon,t}f(z)}{b(A-B)(1-\lambda)G_{\upsilon,t}f(z)-Bz(G_{\upsilon,t}J_{\alpha}f(z))'-G_{\upsilon,t}f(z)}\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}-(n-1)\omega_{n-1}^{\upsilon}(t)a_nz^n}{b(A-B)(1-\lambda)\,z-\sum_{n=2}^{\infty}\left[b(A-B)(1-\lambda)-B(n-1)\omega_{n-1}^{\upsilon}(t)a_nz^n\right]}\right| \qquad (1.12)$$

Then, we need to prove that (1.12) is true. By applyingthe hypothesis (110) and letting $|z|\to 1$, we find that

$$\left|\frac{\sum_{n=2}^{\infty}-(n-1)\omega_{n-1}^{\upsilon}(t)a_nz^n}{b(A-B)(1-\lambda)\,z-\sum_{n=2}^{\infty}\left[b(A-B)(1-\lambda)-B(n-1)\omega_{n-1}^{\upsilon}(t)a_nz^n\right]}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)\omega_{n-1}^{\upsilon}(t)|a_n|}{|b|(A-B)(1-\lambda)-\sum_{n=2}^{\infty}\left[|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)|a_n|}$$

$$\leq \frac{|b|(A-B)(1-\lambda)-\sum_{n=2}^{\infty}\left[|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)|a_n|}{|b|(A-B)(1-\lambda)-\sum_{n=2}^{\infty}\left[|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)|a_n|} \leq 1.$$

Hence, we find that (18) is true. T herefore $f \in M_{\lambda}^{b}(A,B,\upsilon,t)$.
Our assertation in Theorem 1 is sharp for functions of the form

$$f_n(z) = z - \frac{|b|(A-B)(1-\lambda)}{(1+|b(A-B)(1-\lambda)-B|)\omega_{n-1}^{\upsilon}(t)} z^n . \qquad (1.13)$$

3. Distortion Theorems

Theorem 2. If $f \in M_{\lambda}^{b}(A,B,\upsilon,t)$, then

$$r - r^2\left\{\frac{|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\} \leq |f(z)|$$

$$\leq r + r^2\left\{\frac{|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\} \qquad (1.14)$$

with the equality for

$$f_2(z) = z - \frac{|b|(A-B)(1-\lambda)}{(1+|b(A-B)(1-\lambda)-B|)\omega_{n-1}^{\upsilon}(t)} z^2 .$$

Proof. From (1.10),we obtain

$$\sum_{n=2}^{\infty}\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)|a_n| \leq |b|(A-B)(1-\lambda).$$

This implies

$$\sum_{n=2}^{\infty}|a_n| \leq \left\{\frac{|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\} \qquad (1.15)$$

From (1.10) and (1.15) it follows that

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty}|a_n||z|^n \geq r - r^2\sum_{n=2}^{\infty}|a_n| \geq r - r^2\left\{\frac{|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\}$$

In the same manner,

$$|f(z)| \le |z| + \sum_{n=2}^{\infty}|a_n||z|^n \le r + r^2\sum_{n=2}^{\infty}|a_n| \le r + r^2\left\{\frac{|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\}$$

Hence the theorem.

Theorem 3. If $f \in M_{\lambda}^{b}(A,B,\upsilon,t)$ then

$$1-r\left\{\frac{2|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\} \le |f'(z)| \le$$

$$1+r\left\{\frac{2|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\} \qquad (1.16).$$

with equality for

$$f_2(z) = z - \frac{|b|(A-B)(1-\lambda)}{(1+|b(A-B)(1-\lambda)-B|)\omega_{n-1}^{\upsilon}(t)}z^2$$

Proof. By (1.15), we have

$$\sum_{n=2}^{\infty} n|a_n| \le \left\{\frac{2|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\}. \qquad (1.17)$$

From (1.17),it follows that

$$|f'(z)| \ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \ge 1 - r\sum_{n=2}^{\infty} n|a_n|$$

$$\ge 1-r\left\{\frac{2|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\}$$

Similary,

$$|f'(z)| \le 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \le 1 + r\sum_{n=2}^{\infty} n|a_n|$$

$$\le 1+r\left\{\frac{2|b|(A-B)(1-\lambda)}{\left[1+|b(A-B)(1-\lambda)-B|\right]\omega_{n-1}^{\upsilon}(t)}\right\}$$

4.Radii of Close-to-Convexity,Starlikeness and Convexity

A function $f \in T$ is said to be close-to- convex of order $\delta (0 \le \delta < 1)$, if

$$\operatorname{Re}\{f'(z)\} > \delta \,, \qquad (1.18)$$

for all $z \in E$.

A function $f \in T$ is said to be starlike of order $\delta (0 \le \delta < 1)$ if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta. \qquad (1.19)$$

A function $f \in T$ is said to be convex of order $\delta(0 \le \delta < 1)$ if and only if $zf'(z)$ is starlike of order δ that is if

$$\mathrm{Re}\left\{1+\frac{zf'(z)}{f(z)}\right\} > \delta. \qquad (1.20)$$

Theorem 4. If $f \in M_{\lambda}^{b}(A,B,\alpha)$, then f is close-to-convexity of order δ in $|z_1| < r_1(A,B,b,\alpha,\delta,\lambda)$, where

$$r_1(A,B,b,\alpha,\delta,\lambda) = \inf_{n\ge 2}\left[\frac{(1-\delta)((\mathrm{n}-1)+|b(A-B)(1-\lambda)-B(\mathrm{n}-1)|)\omega_{n-1}^{\upsilon}(t)}{n|b|(A-B)}\right]^{1/n}$$

The result is sharp for the function $f_n(z)$ given by (1.13).

Proof. It is sufficient to sufficient to show that

$$|f'(z)-1| \le \sum_{n=2}^{\infty} n|a_n|z^n \le 1-\delta. \qquad (1.21)$$

By (1.10), we have

$$\sum_{n=2}^{\infty}\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)|a_n| \le |b|(A-B)(1-\lambda) \qquad (1.22)$$

observing that (1.21) is true , for fixed n, if

$$\frac{n|z^n|}{1-\delta} \le \frac{\left[(n-1)+|b(A-B)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)}{|b|(A-B)(1-\lambda)} \qquad (1.23)$$

solving (1.23) for $|z|$,we obtain

$$|z| \le \left\{\frac{(1-\delta)\left[(n-1)+|b(\mathrm{A}-B)(1-\lambda)-\mathrm{B}(\mathrm{n}-1)|\right]\omega_{n-1}^{\upsilon}(t)}{n|b|(A-B)(1-\lambda)}\right\}^{1/n}.$$

Theorem 5. If $f \in M_{\lambda}^{b}(A,B,\upsilon,t)$, then f is starlike of order δ in $r_2(A,B,b,\alpha,\delta,\lambda)$ where

$$r_2(A,B,b,\alpha,\delta,\lambda) = \inf_{n\ge 2}\left\{\frac{(1-\delta)\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)}{|b|(n+1-\delta)(A-B)(1-\lambda)}\right\}^{1/n}$$

The result is sharp for the function $f_n(z)$ is given by (1.13).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum_{n=2}^{\infty}(n+2)a_n z^n}{1-\sum_{n=2}^{\infty}a_n z^n} \le 1-\delta\ . \tag{1.24}$$

We see from (1.22) that (1.24) is true if

$$\frac{(n+1-\delta)|z|^n}{1-\delta} \le \frac{\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)}{|b|(A-B)(1-\lambda)} \tag{1.25}$$

solving (1.25) for $|z|$, we obtain

$$|z| \le \left\{\frac{(1-\delta)\left[(n-1)+|b(\mathrm{A}-B)(1-\lambda)-\mathrm{B(n-1)}|\right]\omega_{n-1}^{\upsilon}(t)}{|b|(\mathrm{n}+1-\delta)(A-B)(1-\lambda)}\right\}^{1/n}.$$

Hence the theorem proved.

Theorem 6.If $f \in M_{\lambda}^{b}(A,B,\upsilon,t)$ **,then** f **is convex of order** δ **in** $|z| < r_3(A,B,b,\alpha,\lambda)$ where

$$r_3(A,B,b,\alpha,\lambda) = \inf_{n\ge 2}\left\{\frac{(1-\delta)\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)}{n|b|(n-\delta)(A-B)(1-\lambda)}\right\}^{1/n}$$

The result is sharp for the function $f_n(z)$ is given by (1.13).

Proof. We must show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\sum_{n=2}^{\infty}n(n-1)a_n z^n}{1-\sum_{n=2}^{\infty}na_n z^n} \le 1-\delta \tag{1.26}$$

From(1.22) ,we see that (1.26) is true if

$$\frac{\mathrm{n}(n-\delta)|z|^n}{1-\delta} \le \frac{\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)}{|b|(A-B)(1-\lambda)} \tag{1.27}$$

Solving (1.27) for $|z|$, we obtain

$$|z| \le \left\{\frac{(1-\delta)\left[(n-1)+|b(\mathrm{A}-B)(1-\lambda)-\mathrm{B(n-1)}|\ \right]\omega_{n-1}^{\upsilon}(t)}{n|b|(\mathrm{n}-\delta)(A-B)(1-\lambda)}\right\}^{1/n}$$

Hence the theorem is proved.

5.Convex Linear Combination

We give the result of convex linear combinations as follows:

Theorem 7. Let

$$f_1(z) = z \qquad (1.28)$$

$$f_n(z) = z - \frac{|b|(A-B)(1-\lambda)}{((\mathrm{n}-1)+|b(A-B)(1-\lambda)-B(n-1)|)\omega_{n-1}^{\upsilon}(t)} z^n, \quad n \geq 2 \qquad (1.29).$$

Then $f \in M_{\lambda}^{b}(A,B,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) \qquad (1.30)$$

$$\lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1.$$

Proof. From (1.30) , it is easy to see that

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) \qquad (1.31)$$

$$= z - \sum_{n=2}^{\infty} \frac{|b|(A-B)(1-\lambda)}{((\mathrm{n}-1)+|b(A-B)(1-\lambda)-B(n-1)|)\omega_{n-1}^{\upsilon}(t)} z^n$$

Since

$$\sum_{n=2}^{\infty} \frac{\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)}{|b|(A-B)(1-\lambda)}$$

$$\times \frac{|b|(A-B)(1-\lambda)\lambda_n}{((\mathrm{n}-1)+|b(A-B)(1-\lambda)-B(n-1)|)\omega_{n-1}^{\upsilon}(t)} \sum_{n=2}^{\infty} \lambda_n = 1-\lambda_1 \leq 1.$$

It follows from Theorem 1 that the function $f \in M_{\lambda}^{b}(A,B,\alpha)$.
Conversely , let us suppose that $f \in M_{\lambda}^{b}(A,B,\upsilon,t)$.

Since

$$|a_n| \leq \frac{|b|(A-B)(1-\lambda)}{((\mathrm{n}-1)+|b(A-B)(1-\lambda)-B(n-1)|)\omega_{n-1}^{\upsilon}(t)}, \ (n \geq 2).$$

Setting

$$\lambda_n = \frac{\left[(n-1)+|b(A-B)(1-\lambda)-B(n-1)|\right]\omega_{n-1}^{\upsilon}(t)}{|b|(A-B)(1-\lambda)} a_n, \quad (n \geq 2)$$

And $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$.

It follows that $f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z)$ .This completes the proof of the theorem.

Theorem 8. The class $M_{\lambda}^{b}(A,B,\upsilon,t)$ is closed under convex linear combinations.
Proof. Suppose the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n,\quad (a_{n,j}\geq 0,\ j=1,2;z\in E) \tag{1.32}$$

Are in the class $M_\lambda^b(A,B,\alpha)$ .Setting

$$f(z)=\mu f_1(z)+(1-\mu)f_2(z)\ ,(0\leq\mu\leq 1).$$

We find from (1.27) that

$$f(z)=z-\sum_{n=2}^{\infty}\left[\mu a_{n,1}+(1-\mu)a_{n,2}\right]z^n,\quad (0\leq\mu\leq 1).$$

In view of Theorem 1,we have

$$\sum_{n=2}^{\infty}\left[(n-1)+\left|b(A-B)(1-\lambda)-B(n-1)\right|\right]\omega_{n-1}^{\upsilon}(t)\left[\mu a_{n,1}+(1-\mu)a_{n,2}\right]$$

$$=\mu\sum_{n=2}^{\infty}\left[(n-1)+\left|b(A-B)(1-\lambda)-B(n-1)\right|\right]\omega_{n-1}^{\upsilon}(t)\,\mathrm{a}_{n,1}$$

$$+(1-\mu)\sum_{n=2}^{\infty}\left[(n-1)+\left|b(A-B)(1-\lambda)-B(n-1)\right|\right]\omega_{n-1}^{\upsilon}(t)\,\mathrm{a}_{n,2}$$

$$\leq\mu|b|(A-B)(1-\lambda)+(1-\mu)|b|(A-B)(1-\lambda)=|b|(A-B)(1-\lambda)$$

This completes the proof of the theorem.

References

1. P.L. Duren, Univalent Functions, A series of Comprehensive Studies in Mathematics, vol.259,Springer, New York 1983.
2. J.E. Littlewood, On inequalities in the theory of functions, Proc. London Math. Soc., 23(2), 481-519,1925.
3. G. Murugusundarmoorthy and N. Magesh, Certain sub-classes of starlike functions of complex order involving generalized hypergeometric functions. Int. J. Math. Math. Sci., art ID 178605, 12, 2010.
4. C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, G¨ottingen, 1975.
5. G. Schober, Univalent Functions, Selected topics Lecture Nptes in Mathematics vol. 478, Springer, New York, 1975.
6. H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51, 109-116, 1975.
7. H. Silverman, A survey with open problems on univalent functions whose coefficient are negative, Rocky Mountain J. Math., 21(3), 1099-1125, 1991.
8. H.Silverman, Integral means for univalent functions with negative coefficient, Houston J. Math., 23(1), 169-174, 1997.
9. M. Sobczak-Knec and P. Zaprawa, Covering domains for classes of functions with real coefficients, Complex var. Elliptic Equ. 52(6), 519-535, 2007.
10. J. Szynal, An extension of typically real functions, Ann. Univ. Mariae Curie-Sklodowska, sect. A. 48, 193-201, 1994.
11. P. Zaprawa, M.Figiel, and A.Futa, On coefficients problems for typically real functions related to gegenbauer polynomials, Mediterr. J.Math., 14(2), 1-12, 2017.
12. B. Venkateswarlu, P. Thirupathi Reddy, S. Sridevi, and Sujatha, A certain subclass of analytic functions with Negative coefficients defined by Gegenbauer polynomials, Tatra Mt. Math. Publ. 78 (2021), 73–84.