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EXPONENTIAL SPLINE SOLUTION FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH AN UNCERTAIN BOUNDED PARAMETER

Bhagyamma .D[1], Richa Gupta [2], Ramana Murthy. M.V[3]

[1]Department of Mathematics, Maturi Venkata Subba Rao Engineering College, Nadargul, Hyderabad, Telangana State -501510.

[2]Department of Mathematics, Sarvepally Radhakrishnan University, Bhopal, Madhy Pradesh -462026.

[3]Department of Mathematics & Humanities, Mahatma Gandhi Institute of Technology, Gandipet, Hyderabad.Telangana-500075.

Corresponding author: Richa Gupta

Abstract

Herein, we develop a new numerical method based on absolution for Singularly Perturbed Boundary potential spline and Shishkin mesh discretization to solve singularly per Value Problems with an Uncertain-But Bounded Parameter. We use the Adman decomposition method, a well-known method for solving functional equations nowadays, to solve systems of differential equations of the first order and an ordinary differential equation of any order by converting it into a system of differential of the order one. Some examples are presented to show the method's ability for linear and non-linear systems of differential equations. Theoretical considerations are being discussed, and convergence of the method for these systems is addressed.

Keywords: parabolic equations, absolute errors, fluid dynamics, fluid mechanics.

Introduction

Herein, we consider the following singularly perturbed problem [1];

$$-\varepsilon y'' + \mu f(x)y' + g(x)y = r(x), x \in [a,b]$$

Subject to the following boundary conditions,

$$y(a) - \alpha = y(b) - \beta = 0$$

Where $0 < \varepsilon \ll 1$ and $0 < \mu \ll 1$ are two small perturbation parameters; $f(x), g(x)$ and $r(x)$ are sufficiently smooth functions for$x \in [a,b]$; $a, b, \alpha,$ and β are real constants. In general, the solution $y(x)$ may exhibit two boundary layers of exponential type at both end points$x = a, b$.

Different applications in science and engineering consider these kinds of problems that describe complicated physical and chemical models such as heat transfer problems, Navier-Stokes flow with large Reynolds numbers, chemical reactor theory, convection-diffusion processes, geophysics, aerodynamics, reaction-diffusion processes, quantum mechanics and optimal control, etc [1]. Its solution exhibits two layers at the two endpoints of the domain. The nature of the two-parameter problem was asymptotically examined by [2]. It was found that layer-adapted meshes have been required to obtain a uniformly convergent method no matter how small the perturbation parameter see [3] for more details.

Many numerical methods have been developed for the solution of two layer boundary value problems, such as described in [4-7] for one parameter singularly perturbed boundary value problems and with two small parameters are considered in [8-9], but on a Shishkin-type mesh. Vulanovic [10] considered Shishkin and Bakhvalov meshes but assumed $\mu = \varepsilon^{p+\frac{1}{2}}$with $p > 0$. Dag and Sahin presented a numerical solution of singularly perturbed boundary value problems, using finite element method [11]. Their collocation method was applied with quadratic and cubic B-spline base functions over the geometrically graded mesh of the solution domain. Rashidina and Mohammadi [12] considered the self-adjoint singularly perturbed two-point boundary value problems. Ramadan et al. [13] developed quinticnonpolynomial spline methods for the numerical solution of fourth order two-point polynomial spline methods for the numerical solution of fourth order two-point boundary value problems. A second order monotone numerical method was constructed by Gracia et al. [14].The monotone operator was combined with a piecewise uniform Shishkin mesh. Kadalbajoo and Yadaw [15] presented a B-spline collocation presented a B-spline collocation method for solving a class of two-parameter singularly perturbed boundary value problems. They used B-spline collocation method on piecewise-uniform shown to have a uniform convergence of second order.

The system principally depends on generating a set of consummations of the arbitrary parameter, and also a unique result is defined by carrying out the deterministic solver for each of these consummations. Stein (16) generalized his model which incorporates stochastic goods due to neuronal excitations to handle the distribution of post-synaptic implicit confines and used the Monte-Carlo fashion for approaching the result. David Edwards (17) developed a multi-region FDM fashion for a particular singularly perturbed boundary value problem and this system was grounded on Monte Carlo ways.

The main donation of this paper is to develop a new spline system grounded on a Shishkin mesh discretization for carrying an approximation for the result of two-subcaste boundary value problems. As the anxiety parameter isn't deterministic, thus the interval analysis is considered to estimate the result range. The confirmation of the developed solver will be done by comparing the exact and the approximate results of Hosts' results. The confluence analysis also is presented numerically and shows that the presented system is nearly alternate-order. The paper is organized as follows.

Derivation of Exponential Spline

Discredited the solution region $\Omega = [a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$. Where N is the number of mesh points. Let $h_j = x_{j+1} - x_j, j = 1,2,3, \cdots, N$ be the mesh size and the mesh ratio$\sigma_j = \frac{h_{j+1}}{h_j} > 0, j = 1,2,3, \cdots, N-1$. When $\sigma = 1$ the mesh reduces to a uniform mesh, $h_{j+1} = h_j = h$. The interpolating exponential spline approximation function can be defined as [20]:

$$S(x) = a_j + b_j(x - x_j) + c_j\xi_j\left(\frac{x - x_j}{h_j}\right) + d_j\zeta_j\left(\frac{x - x_j}{h_j}\right), j = 0,1, \cdots, N-1$$

where$\xi_j(x) = 2/\tau^2[\cosh(\tau x) - 1], \zeta_j = 6/\tau^3[\sinh(\tau x) - \tau x], a_j, b_j, c_j, d_j$ are constants and τ is a free parameter such that the non-polynomial spline reduces to usual cubic spline when τ approaches to zero [18], which satisfies the following conditions:

$$S(x_j) \in C^2[a, b],$$

$$S(x_j) = y(x_j), S''(x_j) = M_j.$$

The algebraic manipulations of Equations yield the following expressions:

$$a_j = y_j,$$

$$b_j = \frac{1}{h_j}(y_{j+1} - y_j) + \left(\frac{h_j}{\tau\sinh\tau} - \frac{h_j}{\tau^2}\right)M_{j+1} + \left(\frac{h_j}{\tau^2} - \frac{h_j}{\tau}\coth\tau\right)M_j,$$

$$c_j = \frac{h_j^2}{2}M_j,$$

$$d_j = \frac{\tau h_j^2}{6\sinh\tau}(M_{j+1} - M_j\cosh\tau).$$

From the aspect of the first derivative continuity at the mesh points yields the expression for the determination of $S''(x_j)$where $i = 0, \cdots, N$. We can get the following exponential spline identity relation:

$$y_{j+1} - (1 + \sigma_j)y_j + \sigma_j y_{j-1} = h_j h_{j-1}[\alpha_j M_{j+1} + (1 + \sigma_j)\beta_j M_j + \gamma_j M_{j-1}]$$

where

$$\alpha_j = \frac{1}{\tau_j^2} - \frac{1}{\tau_j\sinh\tau_j}, \beta_j = \frac{\coth\tau_j}{\tau_j} - \frac{1}{\tau_j}$$

$$\gamma_j = \frac{1}{\tau_j^2} + \frac{1}{\tau_j}(\sinh\tau_j - \coth\tau_j\cosh\tau_j), j = 1,2, \cdots$$

Note that, the exponential spline relation is consistent with the standard variable-mesh cubic spline if $\tau \to 0$, hence $\alpha = \gamma = \frac{1}{6}, \beta = \frac{1}{3}$ [19.20].

Mesh Selection Strategy

The domain is $\Omega = [a, b]$ into three sub-domains

$$\Omega_1 = [a, a + \omega_1], \Omega_c = [a + \omega_1, b - \omega_2], \Omega_r = [b - \omega_2, b]$$

where the transition parameters are given by:

$$\omega_1 = \mathrm{m}\left(\frac{1}{4}, \frac{2}{\varphi_1}\ln N\right), \omega_2 = \mathrm{m}\left(\frac{1}{4}, \frac{2}{\varphi_2}\ln N\right)$$

$$\text{and } \varphi_1 = -\max_{x\in[a,b]}\lambda_1(x), \varphi_2 = \min_{x\in[a,b]}\lambda_2(x)$$

where$\lambda_1(x)$ and $\lambda_2(x)$ are two solutions of the characteristic equation:

$$-\varepsilon\lambda^2(x) + \mu f(x)\lambda(x) + g(x) = 0$$

The quantity $\lambda_1 < 0$ describes the boundary layer at $x = a$, while $\lambda_2 > 0$ characterizes the layer at $x = b$, and

$$\lambda_1 = \frac{\mu B - \sqrt{\mu^2 B^2 + 4\varepsilon C}}{2\varepsilon}, \lambda_2 = \frac{\mu B + \sqrt{\mu^2 B^2 + 4\varepsilon C}}{2\varepsilon}, \text{ where } B = \max_{x\in[a,b]} f(x)$$

We take $N/4, N/2$ and $N/4$ mesh points, respectively in $\Omega_l, \Omega_c$and Ω_r. Denote the step sizes in each subinterval by $h_1 = 4\omega_1/N, h_2 = 2(b - a - \omega_1 - \omega_2)/N$and $h_3 = 4\omega_2/N$, respectively. Accordingly, the resulting piecewise-uniform Shishkin mesh is represented by:

$$\check{h} = \begin{cases} h_1 = \dfrac{4\omega_1}{N} & x_j = x_{j-1} + h_1;\ \text{for } j = 1,2,\cdots, N/4, \\ h_2 = \dfrac{2(b - a - \omega_1 - \omega_2)}{N} & x_j = x_{j-1} + h_1;\ \text{for } j = \dfrac{N}{4} + 1,\cdots,3N/4, \\ h_3 = \dfrac{4\omega_2}{N} & x_j = x_{j-1} + h_1;\ \text{for } j = \dfrac{3N}{4},\cdots, N. \end{cases}$$

Interval and Sensitivity Analysis

The interval analysis can be used as a descriptive measure of query in quantitative values. Hence, the anxiety parameter isn't deterministic; the result has to be defined as a range grounded on the interval of the parameter. Accordingly, the upper and lower bounds of the anxiety parameter can be written as

$$\overline{\varepsilon} = \varepsilon^c + \Delta\varepsilon, \underline{\varepsilon} = \varepsilon^c - \Delta\varepsilon$$

Where $\overline{\varepsilon}$ is the upper value, $\underline{\varepsilon}$ is the lower value and ε^c is the central value. Then the fluctuation range of solution could be predictable. Understanding measures can be conducted using different techniques for example One-at-a-Time Sensitivity Measures($\pm SD$), the Sensitivity Index (SI), the Importance Index (II), Differential Sensitivity Analysis (PD), etc. We estimated the sensitivity measures using the following methods One-at-a-Time Sensitivity Measures($\pm$SD), the Sensitivity Index, and the Differential Sensitivity Analysis.

Numerical Example

We consider the following reaction-diffusion problem;

$$-\varepsilon y'' + y = \cos \pi x, x \in [0,1], y(0) = y(1) = 0$$

Solution is given by:

$$Y = c_1 \cos \pi x + c_2 e^{\lambda_1 x} + c_3 e^{-\lambda_2 (1-x)}$$
$$c_1 = \frac{1}{\varepsilon\pi^2 + 1}, c_2 = -c_1 \frac{1 + e^{-\lambda_2}}{1 - e^{\lambda_1 - \lambda_2}}, c_3 = c_1 \frac{1 + e^{\lambda_2}}{1 - e^{\lambda_1 - \lambda_2}}, \lambda_{1,2} = \mp \frac{1}{\sqrt{\varepsilon}}$$

The estimated maximum error E_N and the rate of convergence r_N are computed by the formulas:

$$E_N = \max_{0<j<N} |Y_j - y_j|, r_N = \log_2\|E_N\| - \log_2\|E_{2N}\|$$

Table 1 shows the maximum absolute error and the order of confluence for colorful values of the anxiety parameterε. The results attained using the current system are veritably accurate compared with the logical result and give the order of confluence 2 indeed for small values of ε, and it's shown in Figure 1 that the exact and the approximate results are veritably near. Likewise, since the problem is singularly perturbed its result possesses layers along the boundary of the sphere which is passed in the form of sharp boundary layers in Figure 1 at x =0,1 .

Table 1. Maximum absolute errors and the order of convergence for $= 10^{-k}$.

	$k = 8$		$k = 10$		$k = 12$		$k = 14$	
N	E_N	r_N	E_N	r_N	E_N	r_N	E_N	r_N
2^7	9.49E − 04	1.64	8.64E − 04	1.80	8.57E − 04	1.82	8.56E − 04	1.82
2^8	3.03E − 04	1.55	2.47E − 04	1.85	2.41E − 04	1.89	2.43E − 04	1.89
2^9	1.03E − 04	1.44	6.91E − 05	1.82	6.56E − 05	1.93	6.52E − 05	1.94
2^{10}	3.78E − 05	1.30	1.93E − 05	1.75	1.72E − 05	1.94	1.68E − 05	1.96
2^{11}	1.51E − 05	1.24	5.64E − 06	1.64	4.48E − 06	1.93	4.34E − 06	1.98

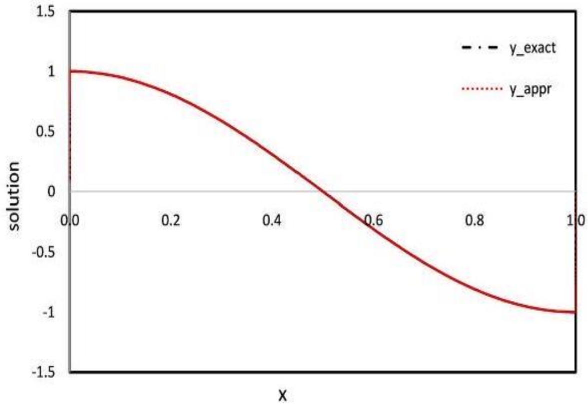

Figure 1. Deterministic Case: exact and approximate solutions for $\varepsilon = 10^{-14}, = 512$.

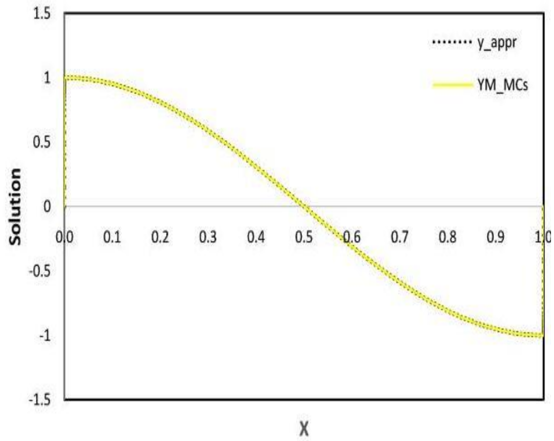

Figure 2. Random Case: Mean of the solution of the proposed method and the MCS results, $\varepsilon = 10^{-14}, = 256$.

Results

$\varepsilon = 10^{-14}, = 256$. The upper, centre and lower solutions of the two methods are very close which shows the accuracy and validation of the proposed method. Further sensitivity measures have been conducted such as the SI method gives an index of value 0.0039%, which is very small. The differential method indicates that the sensitivity coefficient is also very small by value $1.34\text{E} - 15$ at the mid-point of the scheme. Therefore, the solution is not sensitive to the changes in the perturbation parameter.

Conclusion

A numerical system grounded on an exponential spline with Shishkin mesh discretization is combined with an interval analysis perspective to estimate the range of the result for the singularly perturbed two-point boundary value problems with uncertain parameters. Hosts are used to prove the confirmation and the delicacy of the proposed system. Perceptivity analysis has been conducted using different styles and it's a plant that the result isn't sensitive to the anxiety parameter.

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