Some Results on Laplacian Energy of Colored Graphs

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Abstract: This paper obtains a new lower and upper bound for Laplacian energy of a color graph with minimum number of colors. Further, the obtained new bounds are compared with few existing bounds for Laplacian energy of color graph. It is seen that the obtained bounds are better than the existing bounds in the literature.

Key words: Color Laplacian eigenvalues, Color Laplacian energy, Bounds of Color Laplacian energy.

1. Introduction

In recent years, several authors study the properties and results on color energy and color eigenvalues of a graph [1, 2]. Let us consider a simple graph *G = (V, E)*. The adjacency matrix of *G* is the *n x n* matrix and is denoted by *A(G)*, whose entries a_{ij} are given by a_{ij} = 1 if the vertices v_i and v_j are adjacent, a_{ij} = 0 otherwise. The energy, *E(G)* of a graph *G* is the sum of the absolute values of the eigenvalues of *A(G)* ([8]). This energy has a close connection with chemical problems ([15]) and it recently gained attention by mathematicians and mathematical chemists [1, 8, 9, 10]. The study is extended to energy–like quantities for other matrices like adjacency [7, 18], Laplacian [10, 11], signless Laplacian, extended signless Laplacian [4, 5, 6], distance [12], minimum covering [3, 17], label matrix [13] etc., for more than 50 years and this concept is further extended to simple graphs.

A coloring of a graph *G* is obtained by coloring either vertices or edges of a graph. The vertex coloring of a graph involves coloring the vertex of a graph *G* in such a way that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph *G* is called chromatic number and is denoted by χ(G).

Let *G* be a colored graph and if $c(v_i)$ is the color of vertex v_i and the color matrix $A_c(G)$ = $[a_{ij}]$ of *G* is a square matrix of order *n* whose entries are:

$$[a_{ij}] = \begin{cases} 1 & if\ v_i\ and\ v_j\ are\ adjacent \\ & with\ c(v_i) \neq c(v_j) \\ -1 & if\ v_i\ and\ v_j\ are\ non\ adjacent \\ & with\ c(v_i) = c(v_j) \\ 0 & otherwise \end{cases}$$

The eigenvalues of A_c(G) are called color eigenvalues. The energy of a graph with respect to a given coloring is the sum of the absolute values of the color eigenvalues of *G* and it is called as energy of colored graph or color energy of a graph *G*, i.e.,

$E_c(G) = \sum_{i=1}^{n} |\lambda_n|$.

Let *D(G)* be the diagonal matrix of the graph *G*. The Laplacian matrix of graph G is obtained by *L(G) = D(G) – A(G)*. If μ_1, $\mu_2, \ldots, \mu_n$ represent the Laplacian eigenvalues of the graph *G*, then the Laplacian energy is defined as

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|.$$

where *m* is the number of edges and *n* is the number of vertices in *G*. The Laplacian energy has a vast role in chemistry and some basic properties have been found in [10, 19, 20].

The organization of the paper is as follows. Ssection 2, presents some basic definitions which are prerequisites for the study. In section 3, a new lower and upper bounds for color Laplacian energy is obtained. Section 4 deals with the comparison between the exiting bounds and the obtained bound.

2. Color Laplacian Energy

Let *G* be a simple colored graph. The color Laplacian matrix of *G* is defined as $L_c(G) = D(G) - A_c(G)$**.** The eigenvalues $\{\mu_1,\ \mu_2, \ldots, \mu_n\}$ of $L_c(G)$ are called color Laplacian eigenvalues of the graph *G*. The Auxiliary color eigenvalues γ_i is defined as $\gamma_i = \mu_i - \frac{2m}{n}$.

Definition 2.1 [16]: (Color Laplacian matrix)

Let G be a simple colored graph and let $D(G)$ be the diagonal matrix of vertex degrees of the colored graph G. Then $L_c(G) = D(G) - A_c(G)$ is the laplacian matrix of graph. The color Laplacian matrix is obtained by

$$L_c(G) = \begin{cases} d(v_i) & if\ v_i = v_j \\ -1 & if\ v_i\ and\ v_j\ are\ adjacent\ with \\ & c(v_i) \neq c(v_j) \\ 1 & if\ v_i\ and\ v_j\ are\ non\ adjacent\ with \\ & c(v_i) = c(v_j) \\ 0 & otherwise \end{cases}$$

Definition 2.2 [16]: (Color Laplacian energy)

Let G be a colored graph and $\{\mu_1, \mu_2, \ldots, \mu_n\}$ be its eigenvalues. Then the color Laplacian energy of the graph G is

$LE_c(G) = \sum_{i=1}^{n} |\gamma_i|$, where $\gamma_i = \mu_i - \frac{2m}{n}$

i.e., $LE_c(G) = \left|\mu_i - \frac{2m}{n}\right|$.

$LE_c(G)$ is called as chromatic Laplacian energy and is denoted by $LE_\chi(G)$ when a graph G is colored with minimum number of colors.

3. Preliminaries

Some of the important results which are very much needed in proving the main result are mentioned in this section.

Lemma 3.1 [14]

If $\mu_1, \mu_2, \ldots, \mu_n$ are the color Laplacian eigenvalues of graph G, then

$$\sum_{i=1}^{n} \mu_i^2 = 2(\mathrm{m} + m_c') + \sum_{i=1}^{n} d_i^2$$

where m_c' denotes the number of non-adjacent vertices receiving the same color in G.

Property 3.2

If $\mu_1, \mu_2, \ldots, \mu_n$ are the color Laplacian eigenvalues of graph G, then

$$\sum_{i=1}^{n} \mu_i = 2m.$$

Theorem 3.3 (Cauchy – Schwarz inequality)

For all the sequences of a_i and b_i we have,

$$(\sum a_i b_i)^2 = (\sum a_i^2)\ (\sum b_i^2).$$

Theorem 3.4 [1]

Let a_i and b_i , $1 \leq i \leq n$ are positive real numbers, then

$$\sum_{i=1}^{n} b_i^2 + r\,R \sum_{i=1}^{2} a_i^2 \leq (r + R)\left(\sum_{i=1}^{n} a_i b_i\right)$$

where r and R are real constants, so that for each i, $1 \leq i \leq n$, holds $ra_i \leq b_i \leq Ra_i$.

4. Main Results

In this section, a new lower and upper bound for the color Laplacian energy of a colored graph with minimum number of colors are determined.

Theorem 4.1

Let G be a colored graph of order $n > 2$ and size m, then

$$LE_\chi(G) \leq \sqrt{2nM_1 - 4m^2}$$

$where\ M_1 = (\mathrm{m} + m_c') + \frac{1}{2} \sum_{i=1}^{n} d_i^2.$

Proof:

Let us assume $a_i = \left|\mu_i - \frac{2m}{n}\right|$ and $b_i = 1$

By Theorem 3.3, we have

$$\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|^2 \leq \sum_{i=1}^{n}\left(\left|\mu_i - \frac{2m}{n}\right|^2\right)\sum_{i=1}^{n}(1)^2$$

$$\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|^2 \leq \sum_{i=1}^{n}\left(\left|\mu_i - \frac{2m}{n}\right|^2\right) n$$

We have, $LE_\chi(G) = \left|\mu_i - \frac{2m}{n}\right|$ and

$$\left(\left|\mu_i - \frac{2m}{n}\right|^2\right) = \left(\mu_i - \frac{2m}{n}\right)^2$$

$$(LE_\chi(G))^2 = n\sum_{i=1}^{n}\left(\mu_i - \frac{2m}{n}\right)^2$$

$$\leq n\sum_{i=1}^{n}\left[\mu_i^2 + \frac{4m^2}{n^2} - \frac{4m}{n}\mu_i\right]$$

$$\leq n\left[\sum_{i=1}^{n}\mu_i^2 + \sum_{i=1}^{n}\frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n}\mu_i\right]$$

We have,

$$\sum_{i=1}^{n}\mu_i^2 = 2M_1 \ \& \sum_{i=1}^{n}\mu_i = 2m$$

$$\leq n\left[2M_1 + \frac{4m^2}{n} - \frac{8m^2}{n}\right]$$

$$\leq n\left[\frac{2nM_1}{n} + \frac{4m^2}{n} - \frac{8m^2}{n}\right]$$

$$\leq 2nM_1 - 4m^2$$

$$\therefore \ LE_\chi(G) \leq \sqrt{2nM_1 - 4m^2}$$

Theorem 4.2

Let G be a non-complete colored graph of order $n > 2$ and size m and let μ_n and μ_1 be the smallest and the largest eigenvalues respectively, then

$$LE_\chi(G) \geq \frac{\frac{1}{n}[2nM_1 - 4m^2] + n\,|\mu_1|\,|\mu_n|}{(|\mu_1| + |\mu_2|)}$$

$$where \ \ M_1 = (m + m_c') + \frac{1}{2}\sum_{i=1}^{n}d_i^2$$

Proof:

Let us assume $b_i = \left|\mu_i - \frac{2m}{n}\right|$ $a_i = 1$ r = $|\mu_n|$ and

R = $|\mu_1|$

By theorem 3.4 we have

$$\sum_{i=1}^{n}\left(\left|\mu_i - \frac{2m}{n}\right|^2\right) + |\mu_n|\ |\mu_1|\sum_{i=1}^{n}(1)^2 \leq (|\mu_1| + |\mu_n|)\left(\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|\right)$$

We have

$LE_\chi(G) = \left|\mu_i - \frac{2m}{n}\right|$ and $\left(\left|\mu_i - \frac{2m}{n}\right|^2\right) = \left(\mu_i - \frac{2m}{n}\right)^2$

$$\sum_{i=1}^{n}\left(\mu_i-\frac{2m}{n}\right)^2+|\mu_1|\,|\mu_n|\,n\leq\ (|\mu_1|+|\mu_n|)\big(LE_\lambda(G)\big)$$

$$\sum_{i=1}^{n}\left[\mu_i^2+\frac{4m^2}{n^2}-\frac{4m}{n}\mu_i\right]+|\mu_1|\,|\mu_n|\,n\leq\ (|\mu_1|+|\mu_n|)\big(LE_\lambda(G)\big)$$

$$\left[\sum_{i=1}^{n}\mu_i^2+\sum_{i=1}^{n}\frac{4m^2}{n^2}-\frac{4m}{n}\sum_{i=1}^{n}\mu_i\right]+\ |\mu_1|\,|\mu_n|\,n\ \ \leq\ \big(\big|\mu_1+|\mu_n|\big|\big)\big(LE_\lambda(G)\big)$$

$$We\ have\ \sum_{i=1}^{n}\mu_i^2=2M_1\ \ and\ \ \sum_{i=1}^{n}\mu_i=2m$$

$$\left[2M_1+\frac{4m^2}{n^2}\,n-\frac{4m}{n}(2m)\right]+|\mu_1|\,|\mu_n|\,n\ \ \leq\big(\big|\mu_1+|\mu_n|\big|\big)\left(LE_\chi(G)\right)$$

$$\left[2M_1+\frac{4m^2}{n}\ -\frac{8m^2}{n}\right]+\ |\mu_1|\,|\mu_n|\,n\leq\big(\big|\mu_1+|\mu_n|\big|\big)\left(LE_\chi(G)\right)$$

$$\left[\frac{2nM_1}{n}+\frac{4m^2}{n}\ -\frac{8m^2}{n}\right]|\mu_1|\,|\mu_n|\,n\leq\ \big(\big|\mu_1+|\mu_n|\big|\big)\ \left(LE_\chi(G)\right)$$

$$\left[\frac{2nM_1-4m^2}{n}\right]+\ |\mu_1|\,|\mu_n|\,n\ \leq\ \big(\big|\mu_1+|\mu_n|\big|\big)\left(LE_\chi(G)\right)$$

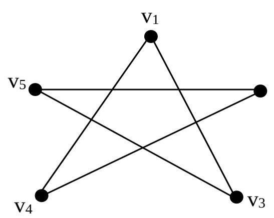
$$\frac{\frac{1}{n}[2nM_1-4m^2]+\ n\ \ |\mu_1|\,|\mu_n|}{\big(\big|\mu_1+|\mu_n|\big|\big)}\leq\ LE_\chi(G)$$

$$\therefore LE_\chi(G)\geq\frac{\frac{1}{n}[2nM_1-4m^2]+\ n\ \ |\mu_1|\,|\mu_n|}{\big(\big|\mu_1+|\mu_n|\big|\big)}$$

5. Comparison Between Newly Obtained Bounds and Existing Bounds

In this section, the comparison is made between the bounds of color Laplacian energy in [12] with the new bounds obtained in this paper. The study shows that the new bounds obtained gives a better result.

Illustration:



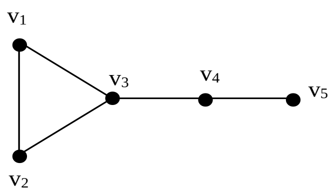
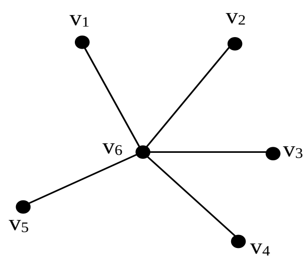


Fig.5.1 Fig. 5.2



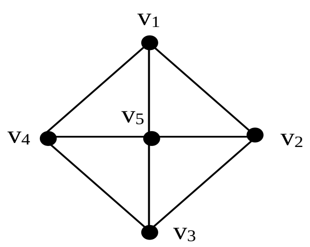


Fig.5.3 Fig.5.4

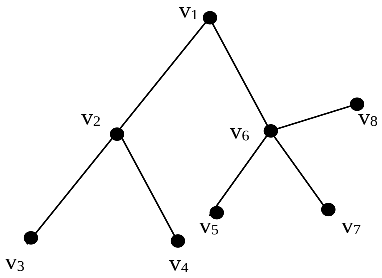


Fig. 5.5

Figure	$2\sqrt{Q} \le LE_c(G) \le 2\sqrt{mQ}$	$\frac{\frac{1}{n}[2nM_1 - 4m^2] + n\ \lvert\mu_1\rvert\ \lvert\mu_n\rvert}{(\lvert\mu_1 + \lvert\mu_n\rvert\rvert)} \le LE_\chi(G \le \sqrt{2nM_1 - 4m^2}$
5.1	$5.2915 \le 7.9516 \le 11.8321$	$4.7998 \le 7.9516 \le 8.3662$
5.2	$5.2915 \le 8.0191 \le 11.8321$	$5.1384 \le 8.0191 \le 8.9437$
5.3	$9.3094 \le 13.32 \le 20.8166$	$8.2917 \le 13.32 \le 14.7196$
5.4	$6.449 \le 7.8722 \le 14.4222$	$6.0641 \le 7.8722 \le 8.5088$
5.5	$10.1980 \le 18.1907 \le 26.9814$	$16.9533 \le 18.1907 \le 21.0481$

6. Conclusion

The new lower and upper bounds for Laplacian energy of a color graph is obtained in this paper and it is compared with existing bounds and found that the obtained bounds give the better results. Further, this work can be extended to get the best bounds in terms of other topological indices and Zagreb indices.

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