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# COMPUTATIONAL ANALYSIS OF TWO-DIMENSIONAL FRACTIONAL DISPERSIVE EQUATIONS

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***Abstract* -** This study presents a computational technique based on Haar Wavelet to find the solutions of fractional dispersive partial differential equations. Fractional dispersive equations have important applications in science and engineering. By applying Haar functions along with its operational matrix, it transforms the fractional dispersive equations to system of linear algebraic equations. These system of equations are then solved by using reasonable number of collocation points. Examples are also given to support the theoretical predictions and then compared with the classical solution of dispersive equations to validate the efficiency of the proposed computational wavelet technique.

## 1. INTRODUCTION

Fractional Calculus has attained significant interest among researchers due to its considerable scope and applications in science and engineering. It belongs to the field of applied mathematics which is as old as the classical one. It covers the derivatives and integrals of non-integer order. Many natural phenomena can be modelled by using fractional derivatives and Integrals. These fractional order models play a vibrant role in unfolding the problems of science and engineering, and have

already been applied in many real-world applications such as control theory (Debnath 2003), viscoelastic systems (Koeller 1984), analysis of electrode processes (Ichise et al. 1971), biological systems (Gómez et al. 2019) and many more.
For numerical approximation of fractional differential equations, numerous approaches have been adopted by researchers (Ali, S.R., et al., 2022). Some of them are Fractional reduced differential transform method (Abuasad et al., 2019), Fractional Laplace Adomian Decomposition method (Mahmood and Arif, 2019 ), Fractional Homotopy Perturbation method (Javeed et al., 2019), Fractional Natural transform (Shah et al., 2018), Fractional Variational iteration method (Durgun and Konuralp 2018), Fractional Integral transform (Demir et al., 2019), Fractional Homotopy Analysis method (Mohamed and Elzaki, 2018), Polynomial based Approximation method (Daşcioğlu et.al., 2019), Wavelet Methods (Singh et al., 2018) and many more.

The basic objective of this paper is to apply Haar wavelet approximation to third order fractional dispersive partial differential equations. Third order dispersive equations have important applications in the theory of water wave, plasma physics and nonlinear optics. Their extensive occurrence in the oceans and solitary waves are of immense importance to oceanographers and geophysicists. Furthermore, the coastal scientists and engineers also utilize the c-noidal wave solutions to examine sedimentation, disintegration of sand particles from beaches, interaction of waves with marine structures nearby coast i.e., piers, jetties, wharfs etc. (Verma et al., 2019).

Many researches have obtained solutions of Fractional order problems through wavelet analysis and approximation (Mustafa, A.R. et al., 2021). Some of these wavelets namely Legendre wavelet (Mohammadi and Cattani, 2018), Chebysheve wavelet (Rafiei et al., 2018), Haar wavelet (Wang et al., 2014; Rehman and Khan 2013; Li and Zhao, 2010) are available in the literature. Among them, Haar wavelet is the simplest one. It has many advantages like simple applicability, orthogonality and compact support. By considering its advantages, Nazir (2019) applied Haar technique on birthmark based identification of software privacy, Zanaty and Ibrahim (2019) used high efficient Haar wavelets for medical imaging compression, Omar et.al (2019) utilized the same wavelet for time fractional reaction sub-diffusion equations, Subrat and chakraverty (2019) studied the vibrant manner of an electromagnetic nanobeam by using Haar wavelet method, Siraj ul haq et.al (2019) numerically solved Sobolev and Benjamin-Bona-Mahony-Burgers equation through Haar wavelet technique.

Previously, Pandey and Mishra (2017) used Homotopy analysis Sumudu transform method to find numerical solutions of fractional third order dispersive equations. Shah et al. (2019) solved third order fractional dispersive equations by Laplace-Adomian decomposition method. Whereas in this study, the authors used Haar wavelet to obtain the approximate solution of the time-fractional third-order dispersive partial differential equations.
The outline of the paper is designed as follows: Section 2 stated some basic theory of fractional calculus. Haar wavelet theory is discussed in section 3. In Section 4, solution process to solve fractional third order dispersive equations are discussed. Section 5 represents the numerical examples to elucidate Haar wavelet. Finally, at the end, conclusion is given in Section 6.

## 2. THEORY OF FRACTIONAL CALCULUS

This section presents few important definitions and properties related to fractional calculus theory for the convenience of the readers.

Many researchers defined fractional derivatives and integrals namely Riemann-Liouville, Atangana-Balenau, Grunwald-Letnikov, and Caputo etc. but the most popular and commonly used definitions on fractional derivative are proposed by Reimann-Liouville and Caputo.

Definition 2.1 Reimann-Liouville fractional integral of a function f ∈ C μ, μ ≥ –1, is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt\,, \quad \alpha > 0, x > 0 \qquad (1)$$

$$J^0 f(x) = f(x)$$

Basic properties of fractional integrals are given below

$For\ f \in C\mu, \mu \geq -1, \alpha, \beta \geq 0, and\ \gamma > 1:$

1. $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x),$ (2)

2. $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$ (3)

3. $J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$ (4)

Definition 2.2 Caputo defined the fractional derivative of the function $f(x)$

$$D^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) \quad (5)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad (6)$$

where

$m-1 < \alpha < m, m \in N, x > 0, f \in C_{-1}^{m}$

For further information, one can consult to ref (Miller and Ross, 1993; Oldham and Spanier, 1974; Podlubny 1999)

## 3. HAAR WAVELET

Haar wavelet functions are defined by

$$h_0(t) = 1 \quad t \in [0,1) \quad (7)$$

$$h_r(t) = \begin{cases} 1 & \frac{k}{m} \le t < \frac{k+0.5}{m} \\ -1 & \frac{k+0.5}{m} \le t < \frac{k+1}{m} \\ 0 & otherwise, \end{cases} \quad (8)$$

where r = 0, 1, 2, ……. m - 1, m = 2s, s ≥ 0, 0 ≤ k ≤ 2s-1, s and k links to integer decomposition of the index r, r = 2s + k - 1, s ≥ 0. Maximum of r is M = 2m =2s +1.

### 3.1 HAAR FUNCTION ANALYSIS

A function $u(t)$ can be expanded into Haar wavelet series by

$$u(t) = \sum_{r=0}^{\infty} d_r h_r(t),$$

$$where \quad d_r = \int_0^1 u(t)\, h_r(t)\, dt \quad (9)$$

Approximating the function u(t) as piece wise constant during each subinterval, Equation 9 can be terminated at fixed terms

$$u(t) \approx \sum_{r=0}^{m-1} d_r h_r(t) = D^T H_m(t), \quad (10)$$

where

$$D = [d_0, d_1, \dots\dots, d_{m-1}]^T,$$

$$H_m(t) = [h_0(t), h_1(t), \dots\dots h_{m-1}(t)]^T, \quad (11)$$

For arbitrary function $u(x,t) \in \mathbb{L}^2([0,1)x[0,1))$ can be approximated into Haar series form as

$$u(x,t) \approx \sum_{r=0}^{m-1}\sum_{s=0}^{m-1} u_{rs} h_r(x) h_s(x) \quad (12)$$

where

$$u_{rs} = \int_0^1 h_r(x)\, h_s(x)\, dx.$$

Equation 12 will be written in matrix form as

$$u(x,t) \approx H_m^T(x) U H_m(t) \quad (13)$$

By using wavelet collocation procedure, we can solve the coefficients $u_{rs}$. The collocation points are defined as :

$$x_r = t_r = \frac{2r-1}{2m}, \quad r = 1,2, \dots \dots\dots, m \quad (14)$$

Discretizing (14) by using eq. (13), to obtain the matrix form of Equation 13

$$D = H^T U H \quad (15)$$

where

$$U = [u_{rs}]_{mxm}, D = [u(x_r, t_s)]_{mxm}$$

Haar matrix $H_m$ is an orthogonal matrix of order m (Wang et al., 2014). The integration of $H_m(t)$ defined in (11) can be approximately converted into haar series with haar wavelet coefficient matrix $P_{m\times m}$.

$$\int_0^t H_m(\tau) d\tau \approx P_{m\times m} H_m(t) \quad (16)$$

where P is called Haar Operational matrix of integration of m$^{th}$ order (Wang et al., 2014).

### 3.2 HAAR OPERATIONAL MATRIX OF FRACTIONAL ORDER INTEGRATION

This section presents Haar wavelet operational matrix of integration for fractional order. The details can be found in (Li and Zhao, 2010). Let $\mathcal{J}^\alpha$ is the fractional integral operator of Haar wavelet, then we get

$$(\mathcal{J}^\alpha H_m)(t) \approx P_{mxm}^\alpha H_m(t) \quad (17)$$

where $P_{mxm}^\alpha$ is known as fractional Haar wavelet operational matrix of integration. We need to define here piece wise Block Pulse Functions as follows,

$$b_r(t) = \begin{cases} 1 & , \frac{r}{m} \le t < \frac{r+1}{m}, \\ 0, & otherwise \end{cases} \quad (18)$$

where

$$r = 0, 1, 2, \dots \dots.(m-1),$$

Also, $b_r(t)$ are orthogonal functions.

As Haar functions are in piecewise form so by extending them into an m-term block pulse functions, we get,

$$H_m(t) = \delta_{mxm} \mathcal{B}_m(t) \quad (19)$$

Where $\mathcal{B}_m(t) \triangleq [b_0(t), b_1(t) \dots \dots. b_i(t) \dots.. b_{m-1}(t)]^T \quad (20)$

$$P^{\alpha}_{mxm} = \delta_{mxm}\mathcal{F}^{\alpha}\delta^{-1}_{mxm} \qquad (21)$$

where

$$\mathcal{F}^{\alpha} = \frac{1}{m^{\alpha}}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (22)$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$

If we take $\alpha = \frac{1}{2}$ and $m = 8$, then fractional Haar operational matrix $P^{\alpha}_{mxm}$ is calculated as

$$P^{0.75}_{8x8} = \begin{bmatrix} 0.752 & -0.220 & -0.155 & -0.082 & -0.110 & -0.058 & -0.044 & -0.037 \\ 0.220 & 0.311 & -0.155 & 0.229 & -0.110 & -0.058 & 0.175 & 0.078 \\ 0.041 & 0.114 & 0.220 & -0.035 & -0.110 & 0.162 & -0.038 & -0.006 \\ 0.077 & -0.077 & 0 & 0.220 & 0 & 0 & -0.110 & 0.162 \\ 0.009 & 0.019 & 0.081 & -0.003 & 0.155 & -0.024 & -0.002 & -0.000 \\ 0.011 & 0.043 & -0.055 & -0.019 & 0 & 0.155 & -0.024 & -0.002 \\ 0.014 & -0.014 & 0 & 0.081 & 0 & 0 & 0.155 & -0.024 \\ 0.027 & -0.027 & 0 & -0.055 & 0 & 0 & 0 & 0.155 \end{bmatrix}$$

## 4. METHODOLOGY

Consider third order dispersive partial differential equation of fractional order

$$\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} + \frac{\partial^{3}v(x,t)}{\partial x^{3}} = q(x,t) \quad t \geq 0,\ 0 < \alpha \leq 1 \qquad (23)$$

$q(x,t)$ is known as source function.

subject to initial-boundary conditions

$$v(x,0) = g(x), \qquad (24)$$

$$v(0,t) = h_0(t),\ v_x(0,t) = h_1(t),\ v_{xx}(0,t) = h_2(t) \qquad (25)$$

By approximating $\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}}$ into 2D Haar wavelet series as

$$\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} = H^{T}_{m}CH_{m} \qquad (26)$$

Integrating (26) and using the initial condition, we get

$$v(x,t) = H^{T}_{m}CP^{\alpha}H_{m} + g(x) \qquad (27)$$

By putting (26) in (23) and integrating with respect to x, we get

$$v(x,t) = M(x,t) - H^{T}_{m}P^{3T}CH_{m} + \frac{x^2}{2}h_2(t) + xh_1(t) + h_0(t) \qquad (28)$$

Equating (27) and (28), we get

$$H^{T}_{m}CP^{\alpha}H_{m} + H^{T}_{m}P^{3T}CH_{m} - N(x,t) = 0 \qquad (29)$$

where

$$N(x,t) = \frac{x^2}{2}h_2(t) + xh_1(t) + h_0(t) - g(x)$$

Solving (29) by using MATLAB, we can get the coefficient matrix.

## 5. Applications

**Problem 1**

Consider the fractional homogenous form of third order dispersive equation

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} + \frac{\partial^{3} v(x,t)}{\partial x^{3}} = 0, \qquad 0 < \alpha \leq 1 \tag{30}$$

subject to conditions

$$v(x,0) = cosx, \tag{31}$$

$$v(0,t) = cost,\; v_x(0,t) = -sint,\; v_{xx}(0,t) = -cost \tag{32}$$

The exact solution for $\alpha = 1$ (Pandey & Mishra 2017) is

$$v(x,t) = \cos(x+t) \tag{33}$$

Numerical results for fractional order at α = 0.5, 0.75 and 1.0 are
computed by the Haar wavelet algorithm. Figure (a), (b) and (c) reflects the approximate solutions for α = 0.5, 0.75 and 1.0 respectively whereas figure (d) shows exact solution.

Moreover, the results given in Table 1 clearly demonstrates that the solutions are in good agreement and are approaching to exact solution for $\alpha = 1$. Also, Figure (i) shows the graphical analysis of problem 1 on different values of $\alpha$.

Table 1 Numerical and exact solution for $v(x,t)$ of Problem 1

| $x_i/t_i$ | Haar Solution | | | $v_{Exact}$ | Absolute Error |
|---|---|---|---|---|---|
| | α = 0.5 | α = 0.75 | α = 1 | | |
| 1/16,1/16 | 0.9922 | 0.9922 | 0.9922 | 0.9922 | 0.0000 |
| 3/16,3/16 | 0.9304 | 0.9306 | 0.9308 | 0.9305 | 3.0 x10-4 |
| 5/16,5/16 | 0.8102 | 0.8107 | 0.8113 | 0.8110 | 3.0 x10-4 |
| 7/16,7/16 | 0.6384 | 0.6401 | 0.6424 | 0.6410 | 1.4 x 10-3 |
| 9/16,9/16 | 0.4253 | 0.4289 | 0.4320 | 0.4312 | 8.0 x 10-4 |
| 11/16,11/16 | 0.1846 | 0.1910 | 0.1974 | 0.1945 | 2.9 x 10-3 |
| 13/16,13/16 | 0.0677 | 0.0585 | 0.0516 | 0.0542 | 2.6 x 10-3 |
| 15/16,15/16 | 0.3133 | 0.3022 | 0.2980 | 0.2995 | 1.5 x 10-3 |

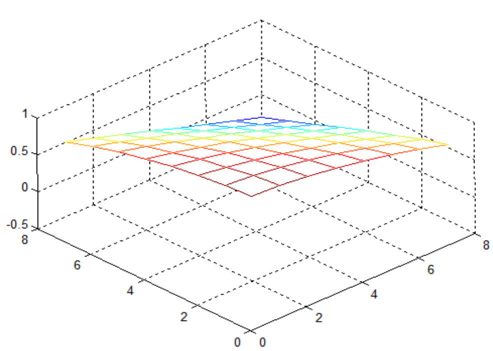


**Fig (a) 3D solution of problem 1 for α =0.5**

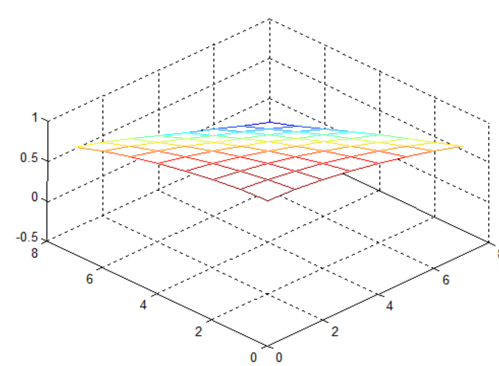


**Fig (b) 3D solution of problem 1 for α = 0.75**

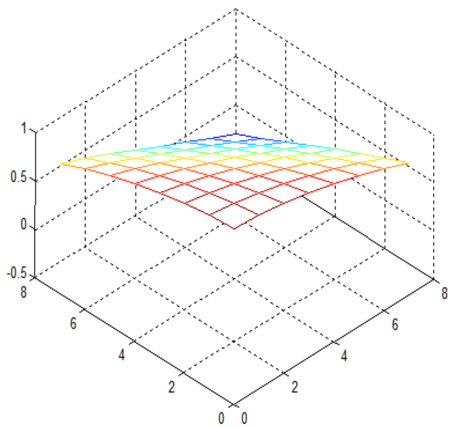



**Fig (c) 3D solution of problem 1 for α = 1.0**

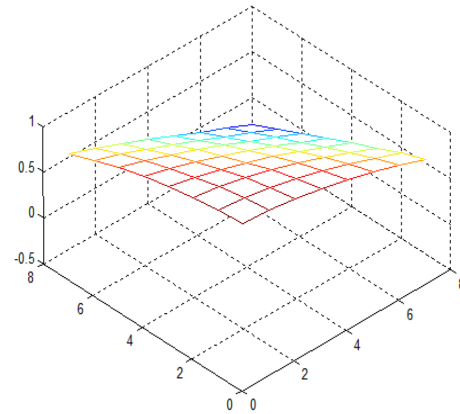



**Fig (d) Exact solution of problem 1 for α =1.0**

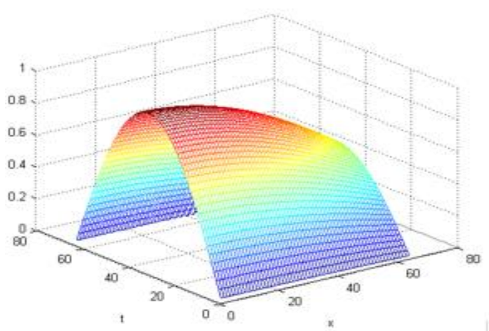



**Fig (e) Haar solution of problem 2 for α = 1.0**

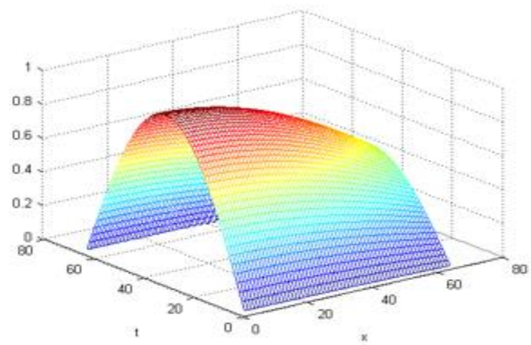



**Fig (f) Exact solution of problem 1 for α =1.0**

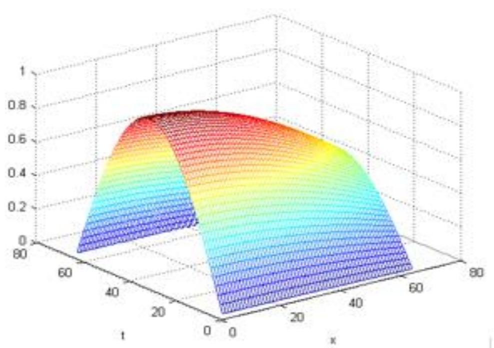



**Fig (g) Haar solution of problem 2 for α =0.5**

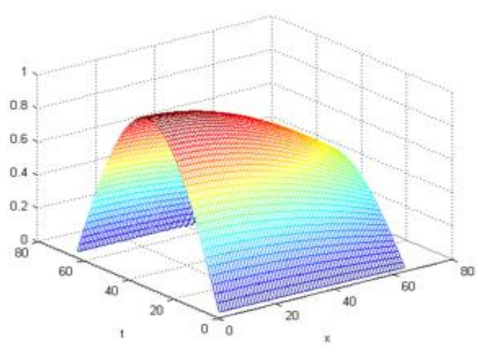
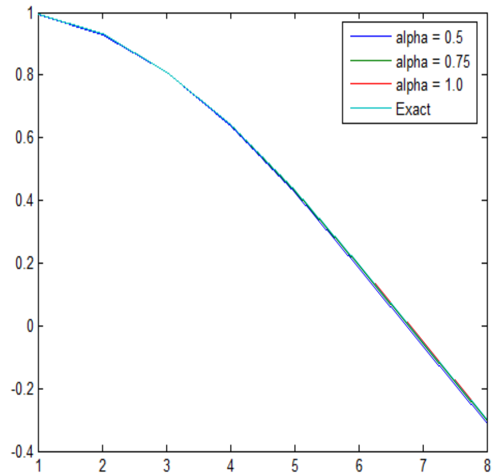



**Fig (h) Haar solution of problem 2 for α =0.75**




**Fig(i) Haar and exact solution for α = 0.5, 0.75, 1.0 of Problem 1**

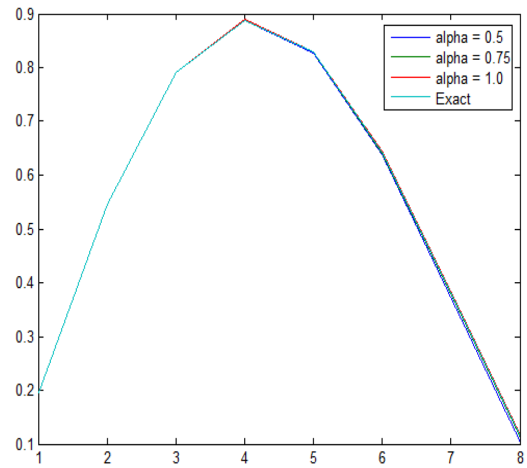



**Fig (j) Haar and exact solution for α = 0.5, 0.75, 1.0 of Problem 2**

**Problem 2**

Consider non-homogenous third order time fractional dispersive partial differential equation

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \frac{\partial^3 v(x,t)}{\partial x^3} = q(x,t) \quad (34)$$

$$q(x,t) = -\sin\pi x \sin t - \pi^3 \cos\pi x \cos t \quad (35)$$

subject to conditions

$$v(x,0) = \sin\pi x \quad (36)$$
$$v(0,t) = 0,\ v_x(0,t) = \pi\cos t,\ v_{xx}(0,t) = 0 \quad (37)$$

The exact solution is

$$v(x,t) = \sin\pi x \cos t \quad (38)$$

In Table 2, the numerical results are given for different values of α. The solutions obtained from fractional cases are in good agreement and approaching to classical order solutions**.** Figure (e) and (f) reflects Haar and exact solution for α = 1 respectively, whereas figure (g) and (h) reflects the Haar solutions for α = 0.5 and 0.75 respectively. Also, Figure (j) shows the graphical analysis of problem 2 on different values of $\alpha$.

Table 2 Numerical and exact solution for $v(x,t)$ of Problem 2

| $x_i/t_i$ | Haar Solution | | | $v_{Exact}$ | Absolute Error |
|---|---|---|---|---|---|
| | α = 0.5 | α = 0.75 | α = 1 | | |
| 1/16,1/16 | 0.1947 | 0.1947 | 0.1947 | 0.1947 | 0.0000 |
| 3/16,3/16 | 0.5458 | 0.5459 | 0.5459 | 0.5458 | 1.0x10-4 |
| 5/16,5/16 | 0.7911 | 0.7912 | 0.7913 | 0.7912 | 1.0x10-4 |
| 7/16,7/16 | 0.8877 | 0.8883 | 0.8891 | 0.8884 | 7.0x10-4 |
| 9/16,9/16 | 0.8275 | 0.8290 | 0.8303 | 0.8297 | 6.0x10-4 |
| 11/16,11/16 | 0.6378 | 0.6409 | 0.6446 | 0.6426 | 2.0x10-3 |
| 13/16,13/16 | 0.3736 | 0.3790 | 0.3838 | 0.3821 | 1.8x10-3 |
| 15/16,15/16 | 0.1031 | 0.1113 | 0.1179 | 0.1155 | 2.4x10-3 |

## 6. CONCLUSION

This paper illustrates the numerical solutions of third order fractional dispersive partial differential equations. The wavelet solutions obtained by Haar technique are promising in finding numerical solutions of fractional order problems arises in the field of science and engineering. The approximations are found for two dispersive fractional partial differential problems for different values of $\alpha$. These solutions converge fast to the exact solutions which can be observed in figures (a-c) and (d-g) for problem 1 and 2 respectively. Consequently, Haar wavelet algorithm gives efficient results and can be implemented to solve other fractional differential equations numerically.

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