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COLINEAR APPROXIMATION IN TERMS OF GENERALIZAL WEIGHTED MODULUS OF SMOOTHESS IN L_P SPACES

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ABSTRACT

Many direct theorems introduced on the unconstrained approximation from 1886 up to now. But very little result introduced about the linearity preserving approximation here we introduce a Jachson type theorem for linearity preserving approximation for linear and positive functions in L_P spaces for $0 < p < 1$ .our results are in terms of the weighted. Modulus of smoothness of the second order

1- Introduction

Let $L_{P[0,1]}$ be the space of all real measurable functions on [0, 1] defined by $L_{P[0,1]}=\{f:[0,1]\to R: \|f\|p < \infty\}$, where $\|f\|_p = \left(\int_a^b |f|^p\right)^{\frac{1}{p}}$ and Ω(0, 1) the class of nonnegative functions $\phi \in L_{P[0,1]}$ which are strictly positive on (0, 1), and such that $\varphi^2$is concave

If $\varphi\in$ Ω(0, 1) and s > 0 defined then terval

$I(\varphi,s) = \{x \in (0,1): 0 \le x - s\varphi(x) < x + s\varphi(x) \le 1\}$

$I(\varphi) = \{s > 0: I(\varphi,s) \ne \varphi\}$ *and* $h_\varphi = (2\varphi(1/2))^{-1}$

For $\varphi \in \Omega$ (0, 1), $f \in L_{P[0,1]}$ and h ∈ (0, hφ] the weighted second-order

modulus is defined by (Ditzian and Totik [11])

$$\omega_2^\varphi(f, h)_p = \sup_{0 \le s \le h} \|f(x - s\varphi(x)) - 2f(x) + f(x + s\varphi(x))\|_{LP(I(\varphi,s))} \quad (1)$$

In $L_{\infty[0,1]}$ we write ω_2^φ (f , h)

The modulus (1) has been used to present estimates in approximation theory. Let us recall some of them. For n ≥ 1 and $f \in L_{P[0,1]}$ the Bernstein operator B_n is defined by

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). x \in [0,1]$$

Note that $B_n(f)$ is a polynomial of degree at most n,Also it easy to see that

$$(B_n(f))(0) = f(0) \quad (B_n(f))(1) = f(1)$$

In [10], Ditzian proved that for α ∈ [0, 1/2] and φ(x) = $(x(1-x))^\alpha$ there exists a constant C_ψ , such that, for $f \in L_{P[0,1]}$ and x ∈ (0, 1)

$$\|f(x) - B_n(f, x)\|_p \le C_\varphi \omega_2^\varphi \left(f. \frac{\sqrt{x(1-x)}}{\sqrt{n}\ \varphi(x)}\right)_p \quad (2)$$

This result unifies the classical estimate for α = 0 (Strukov and Timan [6]) with the norm estimate for α = 1/2 (Ditzian and Totik [11, p. 117]). In [7] Felten proved (2) holds if $\varphi \in \Omega(0, 1)$. On the other hand, in [4] Gavrea et al. verified that

$$\|f - B_n(f)\|_p \le 3\omega_2^\varphi \left(f, \frac{1}{\sqrt{n}}\right)_p$$

for $\varphi(x) = \sqrt{x(1-x)}$. This last estimate improved some others given in [5,3 ,2]. In fact the main result of [4] provides an estimate for positive linear operators that preserve linear functions. The result was improved in [8]: if L: $L_{P[0,1]} \to L_{P[0,1]}$ is a positive linear operator, $f \in L_{P[0,1]}$, $0 < h \le 1/2$ and $x \in (0, 1)$, then

$$\|f(x) - L(f,x)\|_p \le |f(x)||1 - L(e_0,x)| + \frac{|L(e_1-x,x)|}{h\varphi(x)}\omega_1^{\varphi}(f,h)_p + \left(1 + \frac{3}{2}\frac{L((e_1-x)^2,x)}{(h\varphi(x))^2}\right)\omega_2^{\varphi}(f,h)_p$$

In this paper we shall improve and generalize the main results in [4] 0r (3) for φ *in*

$\Omega\,(0,1) = \{\varphi: \varphi(x) \ge 0, x \in (0,1), \emptyset \in L_{P[0,1]}\}$ *we use* a proof similar to that in [7] and $f \in LP[0,1]$ for $0 < P < \infty$

We use $A(\varphi, h) = \{x \in (0,1]: h\varphi(x) < x\}$, $\quad a_h = \inf(A(\varphi, h)$

$$B(\varphi, h) = \{x \in [0,1): h\varphi(x) < 1 - x\} \text{ and } b_h = \sup(B(\varphi, h)),$$

Where $\varphi \in \Omega\,(0,1)$ *and* $h \in (0, h_\varphi)$. For each $x \in [0, b_h]$, the increasing chain $(\{y_n\},\{Z_n\})$ associated to (x,h) is defined as follows .

Let $Z_0 = x, y_1 = x + h\varphi(x)$, *If* $y_1 \ge 1 - h\varphi(1)$ the construction ends in y_1. If $y_1 < 1 - h\varphi(1)$,

and

$$\Delta(f,a,x,b) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) - f(x)$$

2- Auxilary Results

In this section we give the results that we need in our proofs

Lemma 2.1 [1]

Let $\varphi : [0,1] \to R$ be a concave positive function

(1) If $0 \le a < 1$, *Then the function* $M_\varphi(a, o)$ *decreases on* $(a, 1]$.

(2) If $0 \le c < 1$, *Then* $N_\varphi(c, o)$ *increases on* $[0, c)$ *Moreover* *for* $0 \le a \le c \le 1$. max $\{\varphi(a), \varphi(c)\} \le 2\varphi((a + c)/2)$

(3) The limits $\lim_{x\to 1}\frac{\varphi(x)}{x}$ *and* $\lim_{x\to 1}\frac{\varphi(x)}{1-x}$

Exist (finite or infinite)

(4) *If* $c - a \le 2h\varphi(a + c)/2)$ *and* $a \le u < v \le c$, *Then* $v - u \le 2h\varphi(u + v)/2)$.

Lemma 2.2[1]

(1) If $d < b_h \le y_1$, *Then* $\varphi(x) \le 2\varphi(d)$ *and*

(2) If $y_1 < b_h$ *and* $y_1 + h\varphi(y_1) < t$

Then $\varphi(x) \le \varphi(y_1)$ and $\frac{t-x}{y_1-x} \ge 2$

Lemma 2.3[1]

If $(t_2 - t_1) > 2h\varphi(c)$ and $s < 1$

Then $x - sh\varphi(x) < b_h - h\varphi(b_h)$

Lemma 2.4[9]

$\psi\left(\left|\frac{e_1-xe_0}{h}\right|, x\right) = (\frac{3}{2} + \frac{3}{2h^2\varphi(x)}\, L(e_1 - xe_0)^2, x\,)$

Lemma 2.5 [9]

$B_n(e_1 - xe_0)^2, x\,) = \frac{x(1-x)}{n} \le \frac{1}{4n}$

Lemma 2.6

Let $\emptyset \in \Omega\,(0,1)$, $0 \le a \le b \le 1$, $c = \frac{(a+b)}{2}$ *and* $x \in [a,b]$

If $f \in L_{P[0,1]}$ Then

$\|\Delta(\mathrm{f}, \mathrm{a}, \mathrm{c}, \mathrm{b})\|_p \le \omega_2^{\emptyset}\left(\mathrm{g}, \frac{\mathrm{b}-\mathrm{a}}{\emptyset(\mathrm{c})}\right)_p$

Proof :-

Let $g(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) - f(x)$

$$g(a) = \frac{b-a}{b-a} f(a) + \frac{a-a}{b-a} f(b) - f(a) = 0 \quad (1)$$

$$g(b) = \frac{b-b}{b-a} f(a) + \frac{b-a}{b-a} f(b) - f(b) = 0 \quad (2)$$

$$\omega_2^{\emptyset}(f, \delta)_p = \sup_{0 \le h \le \delta} \left\| f\big(x - h\emptyset(x)\big) - 2f(x) + f\big(x + h\emptyset(x)\big) \right\|_{L_{P(I(\varphi,s))}}$$

From (1) and (2) we get $\omega_2^{\emptyset}(g, t)_{L_{P(I(\varphi,s))}} = \omega_2^{\emptyset}(f, t)_{L_{P(I(\varphi,s))}}$

From the defution of g , we can write

$g(c) = -\frac{1}{2}(g(a) - 2g(c) + g(b))$

Let $N_{\emptyset}(c, z) = \frac{\emptyset(Z)}{C-Z}$

Since $N_{\emptyset}$ increasing for $z \in [0,1]$ we get $c < u$

$$\frac{\emptyset(c)}{b-c} = N_{\emptyset}(b, c) \le N_{\emptyset}(b, u) = \frac{\emptyset(u)}{b-u}$$

$$\|g(x)\|_p = \|M\|_p = \|-g(b) + 2g(x) - g(2x-b) - M + g(2x-b)\|_p$$

$$\le \omega_2^{\emptyset}\left(g, \frac{b-a}{\emptyset(a+b)/2}\right)_p$$

Lemma 2.7

If $\emptyset \in \Omega(0,1)$, $h \in (0, h_{\varphi})$, $x \in [a_h, b_h]$ be increasing chain associated to (x,h) of length 1 and $y_1 < 1 - h\emptyset(1)$, $f \in L_{P[0,1]}$ such That

$f\big(x + h\emptyset(x)\big) = 0 = f\big(x + h\emptyset(x)\big)$ then for $t \in [y_1, 1]$

$\|f(t)\|_p \le \frac{3}{2}\frac{t_1}{y_1}\ \omega_2^{\emptyset}(f, h)_p$

Similarly if y_2 exist and $t \in [y_1, y_2]$

Proof

$$\left(\int_0^1 |f(t)|^p\, dx\right)^{\frac{1}{p}} = \left\|\frac{t-x}{y_1-x}\left(\frac{t-y_1}{t-x} f(x) + \frac{y_1-x}{t-x} f(t) - f(y_1)\right) - \frac{t-y_1}{y_1-x} f(x)\right\|_p$$

$$\le \left\|\left(\frac{t-x}{y_1-x} + \frac{1}{2}\ \frac{t-y_1}{y_1-x}\right) f(x - t\emptyset(x) - 2f(x) + f(x + t\emptyset(x)\right\|_p$$

$$\le \left(\frac{t}{y_1} + \frac{1}{2}\frac{t}{y_1}\right) \left(f(x - t\emptyset(x) - 2f(x) + f(x + t\emptyset(x)\right\|_p$$

$$\le \left\|\left(\frac{3}{2}\ \frac{t}{y_1}\right)(f(x - t\emptyset(x) - 2f(x) + f(x + t\emptyset(x)\right\|_p$$

$$= \left(\frac{3}{2}\ \frac{t}{y_1}\right)(\omega_2^{\emptyset}(f, h)_p$$

Lemma 2.8

let $\varphi \in \Omega(0,1)$, $h \in (0, h_{\varphi})$ and $x \in [a_h, b_h]$ be such that the increasing chain associated to (x,h) has length 1 and

$y_1 \ge 1 - h\varphi(1)$

IF $f \in L_{P[0,1]}$ and $f\big(x - h\varphi(x)\big) = 0 = f\big(x + h\varphi(x)\big)$ then

for $t \in [x + h\varphi(x), 1]$

$$\|f\|_p \le \frac{7}{2}\frac{t}{y1}\omega_2^{\varphi}(f, h)_p$$

Proof

let us denote $d = (x + y_1)/2$

case 1:- Assume $b_h \leq d$ Notice that $(y_1 - x) \leq 2(y_1 - b_h)$ and $b_h < y_1$

by using (3) of Lemma 2.1 $t - b_h \leq 2\, h\varphi\, (t + b_h)/2$ then

$\|f(t)\|_p = \left\| \frac{t-b_h}{y_1-b_h} . \left(\frac{t-y_1}{t-b_h} f(b_h) + \frac{y_1-b_h}{t-b_h} f(t) - f(y_1) \right) - \frac{t-y_1}{y_1-b_h} f(b_h) \right\|_p$

$\leq \left\| \frac{1}{y_1-b_h} (2t - b_h - y_1) . f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x)) \right\|_p$

$= \left\| (1 + 2\, \frac{t-y_1}{y_1-b_h}) . f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x)) \right\|_p$

$\leq (1 + 2\frac{t}{y_1}\, f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x))$

$\leq (1 + 2\frac{t}{y_1})\, \omega_2^{\varphi}(f, h)_p = (y_1 + 2\frac{t}{y_1})\, \omega_2^{\varphi}(f, h)_p$

Cas 2: Assume d$<$ $b_h \leq y_1$ by using (1) of Lemma 2.2

$$\|f(t)\|_p \leq \left\| \frac{t-d}{y_1-d} . \left(\frac{t-y_1}{t-d} f(d) + \frac{y_1-d}{t-d} f(t) - f(y_1) \right) - \frac{t-y_1}{y_1-d} f(b_h) \right\|_p$$

$\leq \left\| \frac{1}{y_1-d} (2t - d - y_1) . f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x)) \right\|_p$

$\leq 2\frac{t}{y_1}\quad \omega_2^{\varphi}(f, h)_p$

Cas 3: Assume that $y_1 < b_h$ and t$\leq y_1 + h\varphi(y_1)$as in lemma 2.6 we has

$$\|f(t)\|_p \leq \left\| \frac{t-x}{y_1-x} . \left(\frac{t-y_1}{t-x} f(x) + \frac{y_1-x}{t-x} f(t) - f(y_1) \right) - \frac{t-y_1}{y_1-x} f(x) \right\|_p$$

$\leq \left\| \left(\frac{t-x}{y_1-x} + \frac{1}{2}\, \frac{t-y_1}{y_1-x} \right) f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x)) \right\|_p$

$\leq \left\| \left(\frac{t}{y_1} + \frac{1}{2}\, \frac{t}{y_1} \right) f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x)) \right\|_p$

$\leq (\frac{3}{2}\frac{t}{y_1})\, \omega_2^{\varphi}(f, h)_p$

Cas (4): Assume that $y_1 < b_h$ and $y_1 + h\varphi(y_1) < t$ as in Case (3) we have

$\varphi(x) \leq \varphi(y_1)$ by using (3) of Lemma 2.1 and (2) of Lemma 2.2

$$\|f(t)\|_p \leq \left\| \frac{t-y_1}{h\varphi(y_1)} . \left(1 + \frac{y_{1+h\varphi(y_1)-x}}{y_1-x} + \frac{1}{2}\, \frac{h\varphi(y_1)}{y_1-x} \right) f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x)) \right\|_p$$

$= \left\| \left(2\, \frac{t-y_1}{h\varphi(y_1)} + \frac{3}{2}\, \frac{t-y_1}{y_1-x} \right) f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x)) \right\|_p$

$\leq \left(\frac{2t}{h} + \frac{3}{2}\, \frac{t}{y_1} \right) \omega_2^{\varphi}(f, h)_p$

Sines we get $y_1 < h$

$\|f(t)\|_p \leq \left(\frac{2t}{y_1} + \frac{3}{2}\, \frac{t}{y_1} \right) \omega_2^{\varphi}(f, h)_p$

$\leq \frac{7}{2}\, \frac{t}{y_1}\, \omega_2^{\varphi}(f, h)_p$

Lemma 2.9

Let $\emptyset \in \Omega\, (0,1)$, h$\in (0, h\varphi)$, $x \in [a_h, b_h]$ and $t \in [0,1]$ such that $0 \leq t \leq x - h\varphi(x)$ or $x + h\varphi\,(x) \leq t \leq 1$. If $f \in L_{P[0,1]}$ satisfies $f(x - h\varphi(x) = 0 = f(x + h\varphi(x)$ Then

$$\|f\|_p \leq \frac{7}{2}\, \frac{t}{1-h\,\varphi(1)}\, \omega_2^{\varphi}(f, h)_p$$

Proof

Let $x + h\varphi\ (x) \leq t$. We shall prove this Lemma by induction on the length of the chains

If n=1 Let$(\{y_n\}, \{z_n\})$ increasing chain of (x, h) and of length 1 then $y_1 \geq 1 - h\ \varphi(1)$ or $y_1 < 1 - h\emptyset(1)$, $b_h < z_1$ then using

Lemma 2.7 We get

$\|\mathrm{f(t)}\|_{\mathrm{p}} \leq \frac{3}{2}\frac{t}{y_1}\ \omega_2^{\emptyset}(f, h)_p$

$\leq \frac{7}{2}\frac{t}{y1}\omega_2^{\varphi}(f, h)_p$ where $y_1 < 1 - h\emptyset(1)$

and by Lemma 2.8 $y_1 \geq 1 - h\ \varphi(1)$ We get

$\|f\|_p \leq \frac{7}{2}\frac{t}{y1}\omega_2^{\varphi}(f, h)_p \leq \frac{7}{2}\frac{t}{1-h\ \varphi(1)}\omega_2^{\varphi}(f, h)_p$ for y$\geq 1 - h\ \varphi(1)$

Assume the statement is true for chain of length n and have chain of length $n + 1$

If we eliminate from the chain the point z_0 and y_1 we obtain the chain (z_1 , h) with length n , Let $f \in L_{P[0,1]}$ and $t > x + h\varphi(x)$

If $y_1 < t \leq z_1$ we get the result directly by Lemma 2.7 them assume $t > z_1$ Let P be apolynomial of degree ≤ 1 and interpolate $f$at $y_1$and y_2 let $g = f - p$ using Lemma 2.6 we obtain

$\Delta(f, y_1 , y_2 , t) = \Delta(g, y_1 , y_2 , t)$ by our induction hypothesis we get

$\|\mathrm{f(t)}\|_{\mathrm{p}} \leq \frac{3}{2}\frac{t_1}{y_1}\ \omega_2^{\emptyset}(f, h)_p$

and by Lemma 2.6 we get

$\|\ f\ \|p \leq \omega_2^{\emptyset}\ (\mathrm{g}, \frac{\mathrm{b-a}}{\emptyset(\mathrm{a+b})/2})_{\mathrm{p}}$

$\leq c(p)\ \frac{3}{2}\frac{t_1}{y_1}\ \omega_2^{\emptyset}(f, h)_p$

Wher $c(p)$ is a constant debends on p

3 The Main Results

In this section we introduce our main results in positive linear weighted approximation for functions in $\mathrm{L_p}$ quasi normal spaces for $0 < \mathrm{p} < 1$

Thearem 3.1

If $f \in L_{P[0,1]}$ let $S > 0$, them for

$$\omega_2^{\varphi}(f, \lambda t)_p \leq \lambda 2^{\frac{1}{p}-1}\omega_2^{\varphi}(f, t)_p$$

Proof

Let $0 < S \leq \lambda t$, $\lambda > 1$

Let $x_0 = 0 < x_1 < x_2 < x_3 <\ldots< x_n = 1$, $n \in N$ be a partition for the interval$[0,1]$ such That $|x_i - x_{i+1}| < t$

$\omega_2^{\varphi}(f, \lambda t)_p \leq \left(\int_0^1 |f\ (x - s\varphi(x) - 2f(x) + f(x + s\varphi(x)\ )|^p dx\right)^{1/p}$

$\leq \left(\ \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f\ (x - s\varphi(x) - 2f(x) + f(x + s\varphi(x)\ )|^p dx\right)^{1/p}$

$\leq 2^{\frac{1}{p}-1} \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} |f\ (x - s\varphi(x) - 2f(x) + f(x + s\varphi(x)\ )|^p dx\right)^{1/p}$

$$\leq 2^{\frac{1}{p}-1} \sum_{i=1}^{n} \omega_2^{\varphi}(f, t)_p$$

$$= n\ 2^{\frac{1}{p}-1} \sum_{i=1}^{n} \omega_2^{\varphi}(f, t)_p$$

$$\leq \lambda\ 2^{\frac{1}{p}-1} \sum_{i=1}^{n} \omega_2^{\varphi}(f, t)_p$$

Theorm 3.2

Let $\varphi \in \ \Omega\ (0,1)\ If\ \ f \in \ L_{P[0,1]}$

$$\text{Then } \left\|\Delta(f,t_1,x,t_{2)}\right\|_p \leq 2^{\frac{1}{p}-1}\ \ \frac{7}{2}\left(\frac{2rs+(s+r)}{(r+s)(1-h\varphi(1))}\right)\omega_2^{\varphi}(f,h)_p$$

Proof

Define $s\,,r\ \ and\ \ c\ \ by$

$x-t_1 = sh\varphi(x) \qquad ,t_2 - x = rh\varphi(x)\ and\ \ c = \dfrac{t_1-t_2}{2}$

$If\ s<1 \quad and\ h < \ \dfrac{t_2-t_1}{2\varphi(c)}$

Then $r \geq 1$ In fact $If\ r<1$ Then $s+r<2$

and $[t_1\,,t_2] \subset [\,x-\ rh\varphi(x)\,,x+\ rh\varphi(x)]$

by using (3) of Lemma 2.1

$t_2-t_1 \leq 2h\varphi\ (c)\ \ implies\ \ \ \dfrac{t_2-t_1}{2\varphi(c)} \leq h\ \leq t$

Cas (1): If $(t_2-t_1) \leq 2h\varphi\ (c)$ by Lemma 2.6

$$\|\Delta(\,f\,,t_1,x\,,t_2)\ \|_{L_{P[t_1,t_2]}} = \left\|\ \frac{t_2-x}{t_2-t_1}\,f(t_1) + \frac{x-t_1}{t_2-t_1}\,f(t_2) - f(x)\ \right\|_{L_{P[t_1,t_2]}}$$

$$\leq \omega_2^{\varphi}(f,\tfrac{t_2-t_1}{2\varphi(c)})_{L_{P[t_1,t_2]}} \leq\ \omega_2^{\varphi}(f,h)_{L_{P[t_1,t_2]}}$$

Cas(2): If $(t_2-t_1 > 2h\varphi(c)$ and r$\geq s \geq 1$

we can assume $f(x-h\varphi(x)) = f\big(x+h\varphi(x)\big) = 0.$

By using Lemma 2.9 we have

$$\|f(t_1)\|_p \leq \frac{7}{2}\cdot\ \frac{s}{1-h\varphi(1)}\,\omega_2^{\varphi}(f,h)_{L_P},$$

and

$$\left\|f(t_{)2)}\right\|_p \leq \frac{7}{2}\frac{r}{1-h\varphi(1)}\,\omega_2^{\varphi}(f,h)_{L_P}$$

$$\|\Delta f(t_1,x,t_2)\|_p = \left\|\frac{r}{s+r}f(t_1) + \frac{s}{s+r}f(t_2) - f(x)\right\|_p$$

$$\leq 2^{\frac{1}{p}-1}\left(\tfrac{r}{s+r}\|f(t_1)\|_p + \tfrac{s}{s+r}\|f(t_2)\|_p + \|f(x)\|_p\right)$$

$$\leq 2^{\frac{1}{p}-1}\cdot\tfrac{7}{2}\left(\tfrac{r}{s+r}\ \ \tfrac{s}{1-h\varphi(1)}\omega_2^{\varphi}(f,h)_p + \tfrac{s}{s+r}\ \ \tfrac{r}{1-h\varphi(1)}\omega_2^{\varphi}(f,h)_p + \tfrac{t}{1-h\varphi(1)}\omega_2^{\varphi}(f,h)_p\right)$$

$$\leq 2^{\frac{1}{p}-1}\frac{7}{2}\left(\frac{rs}{(s+r)(1-h\varphi(1)} + \frac{rs}{(s+r)(1-h\varphi(1))} + \frac{1}{\big(1-h\varphi(1)\big)}\right)\omega_2^{\varphi}(f,h)_p$$

$$\leq 2^{\frac{1}{p}-1}\,\tfrac{7}{2}\left(\tfrac{2rs+(s+r)}{(r+s)(1-h\varphi(1))}\right)\omega_2^{\varphi}(f,h)_p$$

Cas(3) by using Lemma 2.3 then $x - sh\varphi < b_h - h\varphi(b_h)$

since $b_h - h\varphi(b_h) \leq 1 - h\varphi(1)$

There exist y such that $t_1 = y - h\varphi(y)$ It is clear that $x<y<b_h$

Now we can assume $f\big(y-h\varphi(y)\big) = f\big(y+h\varphi(y)\big) = 0$

$since\ \ s<1\ then\ \ r \geq 1$

If $t_2 \leq y + h\varphi(y)$ by using Lemma 2.7

$\|f(t_2)\|_p \leq\ \omega_2^{\varphi}(f,h)_p \leq \dfrac{7}{2}\cdot\dfrac{r}{1-h\varphi(1)}\ \omega_2^{\varphi}(f,h)_p$

If $t_2 > y + h\varphi(y)$ by Lemma 2.9 we get

$$\|f(t_2)\|_p \le (\frac{7}{2}.\frac{r}{1-h\varphi(1)})\,\omega_2^{\varphi}(f,h)_p)$$

$$\|\Delta f(t_1,x,t_2)\|_p = \left\|\frac{s}{s+r}f(t_2) - f(x)\right\|_p$$

$$\le 2^{\frac{1}{p}-1}(\tfrac{s}{s+r}\|f(t_2)\|_p + \|f(x)\|_p)$$

$$\le 2^{\frac{1}{p}-1}.\frac{7}{2}(\frac{s}{s+r}\cdot\frac{r}{(1-h\varphi(1)}\omega_2^{\varphi}(f,h)_p + \frac{t}{(1-h\varphi(1)}\,\omega_2^{\varphi}(f,h)_p)$$

$$\le 2^{\frac{1}{p}-1}\frac{7}{2}(\frac{s}{s+r}\frac{1}{(1-h\varphi(1)} + \frac{1}{(1-h\varphi(1)})\,\omega_2^{\varphi}(f,h)_p$$

$$\le 2^{\frac{1}{p}-1}\frac{7}{2}(\frac{2s+r}{(1-h\varphi(1)(s+r)})\,\omega_2^{\varphi}(f,h)_p$$

Remark

Denote $e_0(t) = 1,\ e_1(t) = t\ ,\ e_2(t) = t^2$

Let L be a linear map preseves linearity

$$l((e_1 - xe_0)^2, x) = l(e_2,x) - 2xl(e_1,x) + x^2 = l(e_2,x) - x^2$$

Theorem 3.3

Let$\varphi\epsilon\,\Omega(0,1)$ $l: L_{p[0,1]} \to L_{p[0,1]}$ be a positive linear operator preserves linear functions. If $f \in L_{p[0,1]}$ then

$$\|f - l(f)\|_p \le \frac{3}{2} + \frac{3}{2h^2\varphi^2(x)}(l\,(e_2,x) - x^2))\omega_2^{\varphi}(f,h)_p$$

Proof

Let $\psi(t) = \frac{3}{2} + \frac{3t^2}{2\varphi^2(x)}$

Using Theorem 3.2 we get

$\left\|\Delta(f,t_1,x,t_{2)}\right\|_p \le (\frac{t_2-x}{t_2-t_1}\psi(\frac{x-t_1}{h}) + \frac{x-t_1}{t_2-t_1}\psi(\frac{t_2-x}{h}))\,\omega_2^{\varphi}(f,h)_p$

Then by Lemma 2.5 we get

$$\|f - l(f)\|_p \le c\,l\,(\psi\,\left(\left|\frac{e_1 - xe_0}{h}\right|, x\right)\omega_2^{\varphi}(f,h)_p$$

$$= \tfrac{3}{2} + \tfrac{3}{2h^2\varphi^2(x)}\,l\,((e_1 - xe_0)^2, x))\omega_2^{\varphi}(f,h)_p$$

Theorem 3.4

Let$\varphi\epsilon\,\Omega(0,1)$ and $n \ge 1$ If $f \in L_{p[0,1]}$ then

$$\|f - B_n(f)\|_p \le c(p)\,\omega_2^{\varphi}(f,\frac{\sqrt{x(1-x)}}{\sqrt{n}\,\varphi(x)})_p$$

Where B is Bernstein polynomial

Proof

$B_n((e_1 - xe_0)^2, x) = x\,(1-x)/\,n$

By Theorem 3.3 with $h = \sqrt{x(1-x)}\,/\,(\varphi(x)\sqrt{n}\,)$

$\|f - B_n(f)\|_p \le B_n\,(\psi\,\left(\left|\frac{e_1 - xe_0}{h}\right|, x\right)\omega_2^{\varphi}(f,h)_p$

$$\le c(p)\,\omega_2^{\varphi}(f,\tfrac{\sqrt{x(1-x)}}{\sqrt{n}\,\varphi(x)})_p$$

Conclusion

we introduce a Jachson type theorem for linearity preserving approximation for linear and positive functions in L_P spaces for $0 < p < 1$ .Our results are in terms of the weighted modulus of smoothness of the second order

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