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Double Weights Simultaneous Approximation

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Abstract. Some theorems for simultaneous approximation by algebraic polynomials for continuous functions and its derivatives are introduced. here we introduce a type of double weight modulus of smoothness, and prove a Jackson type estimate interims of it for L_p double weighted simultaneous approximation for $0 < p < 1$. For the prove of the main results we need a double weighted Whitney theorem, we also prove it here, in addition to some properties of the double weighted modulus of smoothness mentioned above.

Keywords.

simultaneous approximation, modulus of smoothness, best approximation.

1. Introduction

1.1 Historical Review

Consider the function analysis one of the important topic in our daily life as he entered into various pure and applied sciences, functional analysis has important applications in physicals, economics And it is concerned with the study of function spaces, where we deal in this research with the space of the L_P and study some theories and properties in this space we deal with the study of approximations in the L_P space with respect to the doubled weights for $0 < p < 1$.

The following is a brief history of the simultaneous approximation.

The following two theorems on simultaneous approximation are known.

Theorem1.1. if f has r continuous differentiable function on $[-1,1]$ and m is a natural number, then here exists an integer n_o depends on r and m s-t for any

$n \geq n_o$ there exists a polynomial P_n of degree $\leq n$ satisfies, for $0 \leq k \leq r$ and $x \in [-1,1]$

$$\left|f^{(k)}(x) - {p_n}^{(k)}(x)\right| \leq c(m,r)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \omega^m\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n}\right) \ldots (1)$$

For $0 \leq k \leq min\{r-m+2, r\}$. The condition $k \leq r-m+2$ cannot be omitted. Theorem1.2. if f has r continuous differentiable function on $[-1,1]$, and m is natural. Then there exists an integer no depending on r and n s-t.

For any $n \geq n_o$ there is a polynomial P_n of degree $\leq n$ s-t.

$\left|f^{(k)}(x) - {p_n}^{(k)}(x)\right| \leq c(m,r)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \omega^m\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n}\right) \ldots (2)$ For $0 \leq k \leq min\{r-m+2, r\}$. Also $k \leq r-m+2$ cannot omitted. [1] 1951 Taman prove Theorem1.1 and Theorem1.2 for $k = 0$ and $m = 1$ In 1955 Gel ford [2] Introduce a proof for Theorem1.1, with $\frac{1}{n}$ instead of $\Delta_n(x)$ and $m = 1$. In 1962 Trigub [3] Prove that Theorem1.1 is valid in the case $m = 1$, and remarked that the same proof is used for $m = 2$. In 1963 Brudnyi [4] Improved Timans result, he proved Theorem1.1 for $k = 0$ and $m \in N$. In 1966 [5]Telyakovskii proved Theorem1.2 in case $m = 1$ and $k = 0$ In 1967, Gopengauz [6] Proved Theorem1.1 in general. In 1975 Devore [7] Proved Theorem1.2 for $m = 2$ and $r = 0$. In 1983 Hinnemann and Gonska [8] Proved Theorem1.3 in the case $m = 2$, $r \geq 0$and $k = 0$ In 1985 they also proved in [9] Theorem1.2 for $k = 0, m \leq r+2$ and $0 \leq k \leq r-m, m \leq$. In1985 yu [9]Proved that Theorem1.2 is not true in the case $k = 0$ and $m \geq r+3$. In 1986. Li [10]And in 1989 Dahlhaus [11] Settled Theorem1.2. here we proved a type of Theorem1.1 and 1.2 for $k \leq r$ as a double weighted simultaneous L_p approximation using spline polynomial, with the aid of double weighted Whitney theorem. also we introduce some properties of the double weighted modulus of smoothness.

1.2 Definitions and Preliminaries

($L_{p,w}$) space:- the space of all function that is $\|f\|_{p,w} < \infty$, define

$$\|f\|_{L_{p,w}}(I) = \left(\int_I |f(u)|^p\, w(u)du\right)^{\frac{1}{p}}, \quad 0 < P < 1.$$

$\boldsymbol{w}$: $[-1,1] \rightarrow \boldsymbol{R}$:- w is called doubling weight if there exists a constant (L) such that $\int_{2I} w(u)du \leq L\int_I w(u)du$, $I \subseteq [-1,1]$.

$\boldsymbol{\omega_{\varphi}^{r}(f\,,t)_{p,w}}$:- is the doubling weight Ditizian Totik modulus of smoothness defined by $\omega_{\varphi}^{r}(f,t) = \sup_{|h|\leq t}\left(\int_I \left|\Delta_{h\ \varphi}^{r}(.)f(x)\right| w(x)dx\right)^{\frac{1}{p}}$

Where the rth symmetric difference is

$$\Delta_h^r\,(f,x,[a,b]) = \begin{cases} \sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i}\ f(x-\frac{rh}{2}+ih)\,, if\ \ x\mp\frac{rh}{2}\in[a,b] \\ 0 \qquad\qquad\qquad\qquad\qquad other\ wise \end{cases}$$

$\varphi(x)=\sqrt{1-x^2}$, $x\ \in[-1,1]$ [12], In our work we use $c(p)$, means constant depending on p , c(p) is different from step to other.

$\boldsymbol{h_j}$:- $|I_j| = (x_{j}-1)-(x_{j})$, $1\leq j\leq n$. [12].

$\boldsymbol{\mathfrak{T}_j}$:- $\begin{cases} I_j\ \cup I_j-1 & if\ 2\leq j\ \leq n \\ I_1 & if\ j=1 \end{cases}$ since $I=[-1,1]$and $I_j\ \subseteq\ I\,, I_j=\lfloor X_j\,, X_j-1\rfloor$. [12]

$\boldsymbol{\pi_n}$:- the space of all polynomials of degree $\leq n$

$\boldsymbol{P_n}$:- the set of polynomials of degree $\leq n$ and $P_n\in\pi_n$

$\boldsymbol{f(x)\ =\ o(g(x))}$:- this means that $g\ (x)\ \neq\ 0$ for sufficiently large x and

$\lim_{x\to\infty}\frac{f(x)}{g(x)}=0.$

$\boldsymbol{J_A}:-\,[a+A(b-a)\,, b-A(b-a)\,]$.

Spline function : we define s to be a continuous piecewise polynomial of degree k, if it is in c[a,b] and if there exists points $\{\zeta_i, i=0,1,\dots,n\}$, satisfying the conditions $a=\zeta_1<\zeta_2<\zeta_3<\cdots<\zeta_n=b$.

2.Auxiliary Results

In this section we introduce the results that we need in our main theorem.

LEMMA 2.1.

If $f\ \in\ L_{p,w}$ and $r\ \in\ N$ for $0\ <\ p\ <\ 1$ we have the inequality

$$\omega^{r}{}_{\varphi}(f,\delta)_{p,w}\leq c(p)\frac{1}{\delta}\int_0^{\delta}\int_{-1}^{1}\left|\Delta_{h\varphi}^{r}(f)\ \right|^{p} w(x)dxdh$$

If $s\geq o$,and $-1+s+2r\delta$, then

$$\sup_{0\leq h\leq\delta}\int_{-1}^{-1+s}|\Delta_h^r\ f(x)|^p w(x)dx$$

$\leq c(p,r)\frac{1}{\delta}\int_0^{\delta}\int_{-1}^{-1+s+r\delta}|\Delta_h^r\,f(x)|^p\ w(x)dxdh$, It is well known

$\Delta_h^r\,f(x)=\sum_{i=1}^{r}(-1)^i\binom{r}{i}\ (\Delta_{\frac{i(t-h)}{r}}^{r}f(x+ih)-\Delta_{h+\frac{i(t-h)}{r}}^{r}f(x)$ then

$$|\Delta_h^r\,f(x)|^p\leq c(p,r)\sum_{i=1}^{r}\left(\left|\Delta_{\frac{i(t-h)}{r}}^{r}f(x+ih)\right|^p+\left|\Delta_{h+\frac{i(t-h)}{r}}^{r}f(x)\right|^p\right)\text{and}$$

$$|\Delta_h^r\,f(x)|^p\leq\frac{c(p,r)}{\delta}\sum_{i=1}^{r}\int_0^{\delta}\left|\Delta_{\frac{i(t-h)}{r}}^{r}f(x+ih)\right|^p\ dt\ \ dt$$

$$+\int_0^{\delta}\left|\Delta_{h+\frac{i(t-h)}{r}}^{r}f(x)\right|^p dt\,, x\in[-1,1],\qquad t,h\in[0,\delta]$$

$$\int_{-1}^{-1+s}|\Delta_h^r f(x)|^p\ w(x)dx$$

$$\leq\frac{c(p,r)}{\delta}\sum_{i=1}^{r}\int_0^{\delta}\int_{-1}^{-1+s}\left|\Delta_{\frac{i(t-h)}{r}}^{r}f(x+ih)\right|^p\ w(x)dxdt$$

$$+\int_0^{\delta}\int_{-1}^{-1+s}\left|\Delta_{h+\frac{i(t-h)}{r}}^{r}f(x)\right|^p\ w(x)dxdt\ \dots(2)$$

In the right hand side of (2) , let $u = x + ih$, $v = \frac{i(t-h)}{r}$ so

$$I_i = \int_0^{\delta}\int_{-1}^{-1+s}\left|\Delta^r_{\frac{i(t-h)}{r}} f(x+ih)\right|^p w(x)dx\,dt$$

$= \int_{-\frac{ih}{r}}^{\frac{i(\delta-h)}{r}}\int_{-1+ih}^{-1+s+ih}|\Delta^r_v f(u)|^p\, w(u)dudv$ it is well known

$\Delta^r_v f(u) = (-1)^r \Delta^r_{-v}\,(u+rv)$

Assume $u - rv = x$, so we get

$$\int_{-\frac{ih}{r}}^{0}\int_{-1+ih}^{-1+s+ih}|\Delta^r_v f(u)|^p\, w(u)dudv$$

$$= \int_0^{-\frac{h}{r}}\int_{-1+ih}^{-1+s+ih}|\Delta^r_v f(u-kv)|^p\, w(u)dudv$$

$$= \int_0^{\frac{ih}{r}}\int_{-1+ih}^{-1+s+ih-rv}|\Delta^r_v f(x)|^p\, w(x)dxdv$$

$$\leq \int_0^{\delta}\int_{-1}^{-1+s+r\delta}|\Delta^r_t f(x)|^p\, w(x)dxdt$$

$$I_i \leq \int_0^{t}\int_{-1}^{-1+s+r\delta}|\Delta^r_t f(x)|^p\, w(x)dxdt$$

Similarly for other integral in (2). So the inequality is proved.

LEMMA 2.2.

If $f \in L_{p,w}$ and $r \in N$ for $0 < p < 1$ we have the inequality

$\sum_{j=1}^{n} \omega^r{}_{\varphi}\left(f, h_j, I_j\right)_{p,w} \leq c(p)\omega^r{}_{\varphi}\left(f, h_j^{-1}\right)_{p,w}.$

Proof.

By using the inequality proved in (lemma1)

$$\omega^r{}_{\varphi}(f,\delta)_{p,w} \leq c(p)\frac{1}{\delta}\int_0^{\delta}\int_{-1}^{1}\left|\Delta^r_{h\varphi}(f)\right|^p w(x)dxdh$$

By use definition of norm and modulus which is

$\|f\|_{L_{p,w}}(I) = (\int_I |f(u)|^p\, w(u)du)^{\frac{1}{p}}$, $0 < P < 1$, and

$\omega^r_{\varphi}(f,t) = \sup_{|h|\leq t} \left\|\Delta^r_{h\varphi}(f)\right\|_{p,w}$

$= \sup_{|h|\leq t} \left(\left|\Delta^r_{h\ \varphi}(.)f(x)\right| w(x)dx\right)^{\frac{1}{p}}$, so we have

$$\omega^r\left(f, h_j, I_j\right)_{p,w} \leq c(p)h_j^{-1}\sup_{|h|\leq t}\int_0^{h_j}\left(\int_I \left|\Delta^r_h(\varphi(f, h_j, I_j)\right|^p\ w(x)dx\right)^{1/p} dh$$

$$\leq c(p)h_j^{-1}\int_0^{h_j^{-1}}\sup_{|h|\leq t}\left\|\Delta^r_{h\varphi}(f)\right\|_{p,w}$$

$$\sum_{j=1}^{n}\omega^r\left(f, h_j, I_j\right)_{p,w} \leq c(p)h_j^{-1}\sum_{j=1}^{n}\int_0^{h_j}\sup_{|h|\leq t}\left\|\Delta^r_{h\varphi}(f)\right\|_{p,w}$$

$\leq c(p)h_j^{-1}\int_0^{h_j}\sup_{|h|\leq t}\left\|\Delta^r_{h\varphi}(f)\right\|_{p,w} \leq c(p)w^r{}_{\varphi}\left(f, h_j^{-1}\right)_{p,w}$, then

$\sum_{j=1}^{n} \omega^r{}_{\varphi}\left(f, h_j, I_j\right)_{p,w} \leq c(p)\omega^r{}_{\varphi}\left(f, h_j^{-1}\right)_{p,w}.$

LEMMA2.3. [14]

$$\Delta_{nh}^{r} f(x)=\sum_{V_1=0}^{n-1}\sum_{V_2=0}^{n-1}\dots\sum_{V_R=0}^{n-1}\Delta_h^r\,(x+V_1h+V_2h+\dots+V_rh)$$

$$\rightarrow \Delta_{nh}^{r} f(x)=\sum_{V=0}^{(n-1)r} A_V^{(r)}\,\Delta_h^r f(x+vh)$$

Where $A_V^{(r)}, V=0,1,\dots,(n-1)r$, are given by the identity

$$(1+t+t^2+\dots+t^{n-1})^r=\sum_{V=0}^{(n-1)r} A_V^{(r)} t^V.$$

LEMMA 2.4.

For any $f\in L_{p,w}(a,b)$, then $\|p_{m-1}(f)\|_{L_{p,w}}\le c(p)\|f\|_{L_{p,w}}$.

Proof.

We suppose that Q is polynomial of degree $\le m-1$, Which interpolate f

At a set of m points in J_A which is superset of the interpellation set for $p_{m-1}(f)$ this immediately impellers that $p_{m-1}(f)=p_{m-1}(Q)$, now using that fact

$\|p-Q\|_{L_{p,w}}\le c(p)\omega_m^r(f,b-a)_{p,w}$, we have

$\|p_{m-1}(f)\|_{L_{p,w}}=\|p_{m-1}(Q)\|_{L_{p,w}}$

$\le c(p)\|Q\|_{L_{p,w}}\le c(p)\|f-Q\|_{L_{p,w}}\le c(p)\|f\|_{L_{p,w}}$

$\le c(p)\omega_m^r(f,b-a)+c(p)\|f\|_{L_{p,w}}\le c(p)\|f\|_{L_{p,w}}$.

LEMMA 2.5. [13]

Let $\mu,\xi\in N$ be such that $\mu\ge 7\xi$, and let $1\le j\le n-1$

be a fixed index. then there exists a polynomial $T_j(x)$ of degree $\le 4\mu n$ such that the following inequalities hold for $x\in[-1,1]$: $|T_j(x)-X_j(x)|\le c(\mu)\psi_j^{\mu}$, and

$|T_j^{(\kappa)}(x)|\le c(\mu)\psi_j^{\mu}h_j^{-\kappa}, 1\le\kappa\le\xi$

Where $X_j(x)=\begin{cases}1, & if\ x\ge x_j\\ 0, & otherwise\end{cases}$.

LEMMA 2.6 (Markov). [14]

Let $p\in P_n$ then $\|p'\|_{L_{p,w}[-1,1]}\le n^2\|p\|_{L_{p,w}[-1,1]}$.

LEMMA 2.7.

Let $n\ge 0, 0<q\le p<1$

Then every polynomial $Q\in P_n$ and every finite interval I_j

Satisfy $(\frac{1}{|I_j|}\int_{I_j}|Q(x)|^q\,w(x)dx)^{\frac{1}{q}}\le\frac{1}{|I_j|}\int_{I_j}|Q(x)|^p\,w(x)dx)^{\frac{1}{p}}\le c(p)\frac{1}{|I_j|}\int_{I_j}|Q(x)|^q\,w(x)dx)^{\frac{1}{q}}$.

Proof.

Let $x_o\in I_j$ such that $|Q(x_o)|=\|Q\|_{L_{p,w}}(I_j)$ Using(lemma 6)

$\|Q'\|_{L_{p,w}}(I_j)\le n^2\|Q\|_{L_{p,w}}(I_j)$

We find that there exists a constant $c(p)=c(q,n)>0$, Such that

$$|Q(X)-Q(x_o)|\le|x-x_o|\|Q'\|_{L_{p,w}}(I_j)\le c(p)\|Q\|_{L_{p,w}}(I_j)\frac{|x-x_o|}{|I_j|}$$

If we set $I_1=\left\{X: X\in I_j, |x-x_o|\le\frac{|I_j|}{2c(p)}\right\}$

Then $|I_1|\ge\frac{|I_j|}{2c(p)}$ and for $x\in I_1$ we have

$\frac{1}{2}\ \|Q\|_{L_{p,w}}\left(I_j\right)\ \leq\ |Q(x_o)| - |Q(x) - Q(x_o)|\ \leq |Q(x)|$, Integrating we find

$$\|Q\|_{L_{p,w}}\left(I_j\right)\ \leq\ 2\left(\frac{1}{|I_j|}\int_{I_j}|Q(x)|^p\ w(x)dx\right)^{\frac{1}{P}}$$

$$\leq c(p)\left(\frac{1}{|I_j|}\int_{I_j}|Q(x)|^p\ w(x)dx\right)^{\frac{1}{P}}.$$

LEMMA 2.8.

Let $r \in N, 0 \leq p \leq 1$

and let I be an arbitrary finite interval. Then for every polynomial

$$Q(x) = \sum_{v=0}^{r} a_v(x - x_0)\ , x_0 \in I$$

Where $\sum_{v=0}^{r}|a_v||I|^v \leq c\left(\frac{1}{|I|}\int_I|Q(x)|^p\ \ w(x)dx\right)^{\frac{1}{P}} \dots (1)$ where $c = c(p, r)$.

Proof.

By translating the interval we can assume that $x_0 = 0$ and $I = [0, b]$

In view of lemma (7) we need to proof (1) for $0 \leq p < 1$

The case I=[0,b] and$0 \leq p < 1$ follows from the case $I = [0,1]$, $0 < p < 1$

by simple change of variables. Finally, the case I=[0,1], $0 < p < 1$

follows from the fact that any two norms in$(r + 1)th$ Dimensional space P_r are equivalent.

LEMMA 2.9.

Let $f \in\ L_{p,w}[a, b], 0 < p < 1$ then there exists a constant c such that

$$\|f - c\|_{L_{p,w}[a,b]} \leq \frac{1}{b - a}\int_a^b\int_a^b|f(x) - f(y)|^p\ w(x)dxdy$$

$$= \frac{2}{b - a}\int_a^{b-a}\int_a^{b-\delta}|f(x + \delta) - f(x)|^p\ w(x)dx\ d\delta\ \leq 2\omega_1\ (f, b - a)_{p,w} \dots (1)$$

Where the constant c can be taken as $c = \frac{1}{b-a}\int_a^b f(\delta)d\delta$ when $P = 1$.

Proof.

Consider the function

$$\emptyset(y) = \int_a^b|f(x) - f(y)|^p\ w(x)dx, y \in [a, b]$$

Cleary , there exists $y_0 \in [a, b]$ such that

$\emptyset(y_0) \leq \frac{1}{b-a}\int_a^b \emptyset(y)dy$,and therefore setting $c = f(y_0)$

We obtain

$$\int_a^b|f(x) - c|^p\ \ w(x)dx\ \leq\ \frac{1}{b - a}\int_a^b\int_a^b|f(x) - f(y)|^p\ \ w(x)dxdy \dots (2)$$

Also, we have

$I = \int_a^b\int_a^b|f(x) - f(y)|^p\ \ w(x)dxdy = \int_a^b\int_x^b|f(x) - f(y)|\ \ w(x)dydx + \int_a^b\int_a^x|f(x) - f(y)|^p\ \ w(x)dxdy$

Substituting$y = x + \delta$and $y = x - \delta$ respectively in the last two integrals and changing respectively the order of integration we get

$$I = \int_a^b\int_a^{b-x}|f(x) - f(x + \delta)|^p\ \ w(x)d\delta dx$$

$$+\int_a^b \int_a^{x-a} |f(x) - f(x-\delta)|^p \ w(x) d\delta dx$$

$$= \int_0^{b-a} \int_a^{b-\delta} |f(x) - f(x+\delta)|^p \, w(x) dx d\delta$$

$$+\int_a^{b-a} \int_{a+1}^{b} |f(x) - f(x-\delta)|^p \ w(x) d\delta dx$$

$$= 2\int_0^{b-a} \int_a^{b-\delta} |f(x+\delta) - f(x)|^p \ w(x) dx d\delta$$

This equality together with (2) implies (1).

LEMMA 2.10.

Let $f \in L_{p,w}(a,b), 0 < p \leq 1,$ and $\omega^r{}_{\varphi}\left(f, \frac{a,b}{r}\right)_{p,w} = 0$

Then there exists a polynomial $Q \in P_{r-1}$ such that $f = Q$ almost everywhere in $[a,b]$.

Proof.

We will prove the lemma by induction with respect to r, in the case $r = 1$ the lemma follows by (lemma 9). Suppose that lemma hold true for some $r \geq 1$,

Without loss of generality we can assume that $[\mathrm{a}, \mathrm{b}] = [0,1]$. suppose that

$$\omega_{r+1}\left(f, \frac{1}{r+1}\right)_{p,w} = \sup_{o \leq h \leq \frac{1}{r+1}} \int_0^{1-(r+1)h} |\Delta_h^{r+1} f(x)|^p \, w(x) dx = 0 \ldots (1)$$

First we shall prove that

$\int_0^{1-th_1-h} \left|\Delta_{\mathrm{h}}^{\mathrm{r}} \Delta_{\mathrm{h}}^1 \mathrm{f(x)}\right|^p w(\mathrm{x}) \mathrm{dx} = 0 \ldots (2), h_1, h \geq 0, rh_1 + h \leq 1$

Indeed, if $h_1 = \alpha h$ and $\alpha = \frac{m}{n}$ with some integers m, n then by (lemma 3) we get

$|\Delta^r_{\left(\frac{m}{n}\right)h} \, \Delta^1_h \, f(x)|^p \leq \sum_{v_1}^{m-1} \sum_{v_2}^{m-1} \ldots \ldots \sum_{v_r}^{m-1} |\Delta^r_{\frac{h}{n}} \, \Delta^1_h \, f\left(x + v_1 \frac{h}{n} + \cdots + v_r \frac{h}{n}\right)|^{\mathrm{p}}$

$$\leq \sum_{\mathrm{v}_1=0}^{\mathrm{m}-1} \sum_{\mathrm{v}_2}^{\mathrm{m}-1} \ldots \ldots \sum_{\mathrm{v}_\mathrm{r}}^{\mathrm{m}-1} \sum_{\mathrm{v}=0}^{\mathrm{m}-1} \left|\Delta^{\mathrm{r}+1}_{\frac{\mathrm{h}}{\mathrm{n}}} \, \mathrm{f}\left(\mathrm{x} + \frac{\mathrm{v}_{1\mathrm{h}}}{\mathrm{n}} + \cdots .. + \frac{\mathrm{v}_\mathrm{r}\mathrm{h}}{\mathrm{n}} + \frac{\mathrm{vh}}{\mathrm{n}}\right)\right|^{\mathrm{p}}$$

Integrating with respect to $x \in \left[0, 1 - \left(\frac{rm}{n+1}\right) h\right]$, and using (1) we conclude that (2) holds true in the case considered. Suppose that $h_1 = \alpha h, \alpha > 0$, an irrational number. Choose a sequence $\{\alpha_i\}_{i=1}^{\infty}$ of rational number. Such that $\alpha_i \to \alpha$ as $i \to \infty$ and $0 < \alpha_i < \alpha$ we have

$|\Delta^r_{\alpha h} \, \Delta^1_h \, f(x)| \leq \left|\Delta^r_{\alpha_i h} \Delta^1_h f(x)\right| + \sum_{v=0}^{r} \binom{r}{v} \{\left|f(x + v_{\alpha h} + h) - f\left(x + v_{\alpha_i h} + h\right)\right| + \left|f(x + v_\alpha h) - f\left(x + v_{\alpha_i} h\right)\right|\}$, and there fore

$$\sup_{h \leq \alpha_i} \int_0^{1-r_\alpha h-h} |\Delta^r_{\alpha h} \Delta^1_h \, f(x)|^p \, w(x) dx \leq \sup_{\mathrm{h} \leq \alpha} \int_0^{1-r_{\alpha_\mathrm{i}} h-h} \left|\Delta^r_{\alpha_\mathrm{i} h} \Delta^1_h \, f(x)\right|^p w(x) dx +$$

$c(r,p) w_1(f, r(\alpha - \alpha_i)h)_{p,w}$

$= c(r,p) w_1(f, r(\alpha - \alpha_i)h)_{p,w} \ldots (3)$

Where we have used that (2) holds true for α_i a rational number since

$\omega_1(f, (\delta)h)_{p,w} \to 0$ as $\delta \to 0$. (3) implies (2)

In view of (2) our induction hypothesis gives that for each h $0 \leq h < 1$, there exists a polynomial $Q_n \in p_{r-1}$ such that $\Delta^1_h \, f(x) = Q_n(x)$ for almost all $x \in [0, 1-h]$, $i.e \; f(x+h) - f(x) = \sum_{r=0}^{r-1} a_r(h) x^r \ldots \ldots (4)$

Almost everywhere in $[0, 1-h]$, we shall prove that each coefficient $a_r(h)$ is continuous function of $h \in [0,1)$.

Let $0 \leq h_1, h_2 < 1$. We apple lemma (8) to the polynomial

$$\sum_{v=0}^{r-1} (a_r(h_1) - a_2(h_2) \, x^r = f(x+h_1) - f(x+h_2)$$

for the interval $I = [0, 1-h], h = max\,\{h_1, h_2\}$. we obtain

$$\sum_{v=0}^{r-1} | a_r(h_1) - a_r(h_2)||1-h|^v$$

$$\leq c(r,p)(\frac{1}{1-h} \sup_{|h|\leq\alpha} \int_0^{1-h} |f(x+h_1) - f(x+h_1)|^p\, w(x)dx)^{\frac{1}{p}}$$

$$\leq c(r,p)\frac{1}{1-h} \quad \omega_1(f, h|h_1 - h_2|)_{p,w}$$

since $\omega_1(f,\delta) \to 0$ as $\delta \to 0$

It follows that $a_v(h)$ is continuous function of $h \in [0,1)$

Applying now an arbitrary $(r+1)th$ difference Δ_t^{r+1} to (4)

As function of h we obtain $\Delta_t^{r+1} f(x+h) = \sum_{v=0}^{r-1}(\Delta_t^{r+1} a_v(h))\, x^v$

For almost all $x \in (0, 1-h-(r+1)t)$ and $t, h \geq 0, h+(r+1)t < 1$

By the fact $[\inf_{Q\in P_{n-1}} \|f-Q\|_{p,w} = 1, f \in L_{pw,}(0,1)]$

It follows that for almost all $x \in [0, 1-(r+1)h, 0 \leq h \leq \frac{1}{r+1}$

We have $\Delta_h^{r+1} f(x) = 0,$ and there for since $a_v(h)$ is continuous function of h we have $\Delta_t^{r+1} a_v(h) = 0, 0 \leq h < 1-(r+1)t, 0 \leq t < \frac{1}{r+1},$

$v = 0,1,\dots,r-1.$

3. The Main Results

Here let us introduce our main theorems.

Theorem3.1.

The double weighted modulus of smoothness $\omega^r{}_\varphi(f,\delta)_{p,w}$. have the following

Properties for$f \in L_{p,w}$, $0 < p < 1, r \geq 1$, we have

(1) $\lim_{\delta\to 0} \omega^r{}_\varphi(f,\delta)_{p,w} = 0.$

Proof.

$$\lim_{\delta\to 0} \omega^r{}_\varphi(f,\delta) = \lim_{\delta\to 0} \sup_{|h|\leq\delta} \left\|\Delta_{h\varphi}^r(f)\right\|_{p,w}$$

$$= \lim_{\delta\to 0}\left[\sup_{|h|\leq\delta}\left(\int_I \left|\Delta_{h\varphi}^r(f)\right| w(x)dx\right)^{\frac{1}{p}}\right]$$

$$\sup_{|h|\leq\delta}\left[\lim_{\delta\to 0}\left(\int_I \left|\sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i} f\left(x - \frac{rh}{2} + ih\right)\right| w(x)dx\right)^{\frac{1}{p}}\right]$$

$$\sup_{|h|\leq\delta}\left[\left(\int_I \lim_{\delta\to 0}\left|\sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i} f\left(x - \frac{rh}{2} + ih\right)\right| w(x)dx\right)^{\frac{1}{p}}\right]$$

$$= \sup_{|h|\leq\delta}\left[\left(\int_I \lim_{\delta\to 0}\left|\sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i} f\left(x - \frac{r\delta}{2} + i\delta\right)\right| w(x)dx\right)^{\frac{1}{p}}\right]$$

$$= \sup_{|h|\leq\delta}\left[\left(\int_I \left|\sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i} f\left(x - \frac{r0}{2} + i0\right)\right| w(x)dx\right)^{\frac{1}{p}}\right] = 0, \text{since } \delta \to 0,$$

Then $\omega^r{}_\varphi(f,\delta)_{p,w} \to 0$, then $\lim_{\delta\to 0} \omega^r{}_\varphi(f,\delta)_{p,w} = 0$

(2) $\omega^r{}_\varphi(f,\delta)_{p,w}$ is a nondecreasing function of δ.

Proof.

Let $\delta_1 < \delta_2$ then

$\omega^r{}_{\varphi}(f,\delta_1)_{p,w} = \sup_{|h|\le\delta_1} \left(\int_I \left|\Delta^r_{h\varphi}(f)\right| w(x)dx\right)^{\frac{1}{p}}$, and

$$\omega^r{}_{\varphi}(f,\delta_2)_{p,w} = \sup_{|h|\le\delta_2} \left(\int_I \left|\Delta^r_{h\varphi}(f)\right| w(x)dx\right)^{\frac{1}{p}}$$

Since $\delta_1 < \delta_2$

$$\sup_{|h|\le\delta_1} \left(\int_I \left|\Delta^r_{h\varphi}(f)\right| w(x)dx\right)^{\frac{1}{p}} < \sup_{|h|\le\delta_2} \left(\int_I \left|\Delta^r_{h\varphi}(f)\right| w(x)dx\right)^{\frac{1}{p}}$$

then$\omega^r{}_{\varphi}(f,\delta_1)_{p,w} < \omega^r{}_{\varphi}(f,\delta_2)_{p,w}$, *so* $\omega^r{}_{\varphi}(f,\delta)_{p,w}$ is a nondecreasing function of δ.

(3) $\omega^r{}_{\varphi}(\alpha f + \beta g,\delta)_{p,w}$

$\le c(p)\left(|\alpha|^p \omega^r{}_{\varphi}(f,\delta)_{p,w} + |\beta|^p \omega^r{}_{\varphi}(g,\delta)_{p,w}\right)$.

Proof.

$$\omega^r{}_{\varphi}(\alpha f + \beta g,\delta)_{p,w} = \sup_{|h|\le\delta} \left\|\Delta^r_{h\varphi}(\alpha f) + \Delta^r_{h\varphi}(\beta g)\right\|_{p,w}$$

$$\le c(p)\left(\sup_{|h|\le\delta} \left\|\Delta^r_{h\varphi}(\alpha f)\right\|_{p,w} + \sup_{|h|\le\delta} \left\|\Delta^r_{h\varphi}(\beta g)\right\|_{p,w}\right)$$

$$\le c(p)[\sup_{|h|\le\delta} \left(\int_I \left|\Delta^r_{h\ \varphi}(\alpha f)\right|^p w(x)dx\right) + \sup_{|h|\le\delta} \left(\int_I \left|\Delta^r_{h\ \varphi}(\beta g)\right|^p w(x)dx\right)]$$

$$\le c(p)[|\alpha|^p(\sup_{|h|\le\delta} \int_I \left|\Delta^r_{h\ \varphi}(f)\right|^p w(x)dx) + |\beta|^p \sup_{|h|\le\delta} \int_I \left|\Delta^r_{h\ \varphi}(\beta g)\right|^p w(x)dx]$$

$$\le c(p)[|\alpha|^p \sup_{|h|\le\delta} \left\|\Delta^r_{h\varphi}(f)\right\|_{p,w} + |\beta|^p \sup_{|h|\le\delta} \left\|\Delta^r_{h\varphi}(g)\right\|_{p,w}]$$

$$\le c(p)\left(|\alpha|^p w^r{}_{\varphi}(f,\delta)_{p,w} + |\beta|^p w^r{}_{\varphi}(g,\delta)_{p,w}\right).$$

(4)$\omega^r{}_{\varphi}(f,n\delta)_{p,w} \le c(p,r)\, n^{r-1+\frac{1}{p}}\, \omega^r{}_{\varphi}(f,\delta)_{p,w}$

And there for $\omega^r{}_{\varphi}(f,\lambda\delta)_{p,w} \le c(p,r)(\lambda+1)^{r-1+\frac{1}{p}} \omega^r{}_{\varphi}(f,\delta)_{p,w}$, $\lambda \ge 0$.

Proof.

By using lemma (3) we get

$A_V^{(r+1)} \le n \max_{V} A_V^{(r)} \le n^r, V = 0,1,\ldots,(n-1)(r+1)$

Now

$$\sup_{|h|\le\delta} \left(\int_0^{r-1+\frac{1}{p}} \left|\Delta^r_{hn\ \varphi} f(x)\right|^p w(x)dx\right)$$

$$\le \sum_{V=0}^{(n-1)r} \left(A_V^{(r)}\right) \sup_{|h|\le\delta} (\int_0^{r-1+\frac{1}{p}} \left|\Delta^r_{hn\ \varphi} f(x)\right|^p w(x)dx$$

$\le c(p,r) n \omega^r{}_{\varphi}(f,\delta)_{p,w}\ \ 0 < h < \delta$.

(5) if $f \in L_{p,w}$ and $\omega^r{}_{\varphi}(f,\delta)_{p,w} = O\left(\delta^{r-1+\frac{1}{p}}\right)$ then

F is a polynomial of degree $\mathrm{r}-1$ for almost all $x \in (a,b)$.

Proof.

From property (4) there follows immediately the following estimate

$\omega^r{}_{\varphi}(f,\delta_2)_{p,w}/\delta_2^{r-1+\frac{1}{p}} \le c(p,r)\omega^r{}_{\varphi}(f,\delta_1)_{p,w}/\delta_1^{r-1+\frac{1}{p}}$,$0 < \delta_1 < \delta_2$

If $\omega^r{}_{\varphi}(f,\delta)_{p,w} = O\left(\delta^{r-1+\frac{1}{p}}\right)$Then

$\omega^r{}_{\varphi}(f,\delta)_{p,w} = 0, \delta > 0$

And according to lemma (10) we conclude that $f \in L_{p,w}$, $0 < p \leq 1$, and

$$\omega^r{}_{\varphi}\left(f, \frac{b-a}{r}\right)_{p,w} = \mathbf{0}$$

Then there exists polynomial $f \in P_{r-1}$. Such that $f = Q$ almost everywhere in [a,b]

we conclude that f coincides with a polynomial of degree $r-1$. note that we have for

$$f(x) = \begin{cases} 0, & x \in (-1,0) \\ x^{r-1}, & x \in (1,0) \end{cases}$$

The estimate $\omega^r{}_{\varphi}(f,\delta)_{p,w} = O\left(\delta^{r-1+\frac{1}{p}}\right)$

We remark that there is no upper estimate of

$\omega^r{}_{\varphi}(f,\delta)_{p,w}$ By $\omega_{r-1}(f',\delta)_{p,w}$ or $\left\|f^{(r)}\right\|_{p,w}$ When $f' \in L_{p,w}$ of $f^{(r)} \in L_{p,w}$

Respectively, in the case $0 < p < 1$ Indeed, consider the function

$$\varphi(x) = \begin{cases} 0 & x \in [-1,0] \\ \epsilon^{-1}x & x \in [0,\epsilon] \\ 1 & x \in [0,1] \end{cases}$$

Where $\epsilon > 0$

Is sufficiently small. It is readily seen that $\omega^r{}_{\varphi}(\varphi_\epsilon,\delta)_{p,w} > 0$, and

$\|\varphi_\epsilon^r\|_{p,w} = \epsilon^{\frac{1}{p}-1} \to 0$ as $\epsilon \to 0$

This fact is central importance and will influence fundamentally our further discussion.

THEOREM 3.2.

Let $f \in L_{p,w}(a,b)$, $0 < p < 1, m \epsilon N$ and let $p_{m-1}(f) \epsilon\, \pi_{m-1}$. Interpolate f at m points in $J_A = [a + A(b-a), b - A(b-a)]$where $A < 1/2$ is strictly positive constant, then

$\|f - p_{m-1}(f)\|_{L_{p,w}} \leq c(p)\omega_m^r(f, b-a)_{p,w}$.

Proof.

For any $f \in L_{p,w}[a,b]$ using (lemma 4)we get

$\|p_{m-1}(f)\|_{L_{p,w}} \leq c(p)\|f\|_{L_{p,w}}$

And let $q_{m-1}\, \pi_{m-1}$ be any polynomial in π_{m-1} such that

$\|f - q_{m-1}\|_{L_{p,w}[a,b]} \leq c(p)\omega_m^r(f, b-a)_{p,w}$

Then taking into account that $f - q_{m-1} \in L_{p,w}(a,b)$

We have

$\|f - p_{m-1}(f)\|_{L_{p,w}[a,b]} = \|f - q_{m-1} - p_{m-1}(f - q_{m-1})\|_{L_{p,w}[a,b]}$

$\leq c(p)\|f - q_{m-1}\|_{L_{p,w}[a,b]} + c(p)\|p_{m-1}(f - q_{m-1})\|_{L_{p,w}[a,b]}$

Since

$\|p_{m-1}(f)\|_{L_{p,w}[a,b]} \leq c(p)\|f\|_{L_{p,w}[a,b]}$

$\leq c(p)\|f - q_{m-1}\|_{L_{p,w}[a,b]} \leq c(p)\omega_m^r(f, b-a)_{p,w}$.

COROLLARY 3.3.

As a corollary of theorem (2) we get the following result,

Let $r \in N$ and $f = f(x)$ be such that $f^{(r-1)}$ and $f^{(r)} \in L_{p,w}[a,b]$, $0 < p < 1$, then for every $m \in N$

A linear operator $Q_{m+r-1}(f,[a,b]): L_{p,w}[a,b] \to \pi_{m+r-1}$

$$\left\|f^{(n)} - Q_{m+r-1}^{(n)}\,(f,[a,b]\right\|_{L_{p,w}[a,b]} \leq c(p)\omega^{m+r-n}\,(f^{(n)}\,,\frac{b-a}{2(m+r)})_{p,w}$$

For $n = 0,1,2,\dots,r$.

THEOREM 3.4.

Let $m,r \in N$ and $f = f(x)$ such that $f^{(r)} \in L_{p,w}[-1,1]$

$o \leq p < 1$, then for any $n \geq m+r-1$ there exists a linear operator

$p_n(f): L_{p,w}[-1,1] \to \pi_n$ such that

(1) $\left\|\frac{f^{(k)}(x)-p_n^{(k)}(f,x)}{I_j{}^{rk-k}(x)}\right\|_{p,w} \leq c(p)\omega_{\varphi}^{m+r-rk}\left(f^{(rk)},\frac{1}{n}\right)$

For $k = 0,1,\dots,r$ and any integer rk satisfying $k < rk < r$

Moreover, for $k \geq m+r$ and any integer $r^{\sim}$, $0 \leq r^{\sim} \leq r$

The following inequality holds

(2) $\left\|I_j(x)^{k-r^{\sim}} p_n^{(k)}\,(f,x)\right\|_{p,w} \leq c(p)\omega_{\varphi}^{m+r-rk}\,(f^{(rk)}\,,\frac{1}{n})_{p,w}$.

Proof.

We approximate f and its derivatives $f^{(k)}$ by the spline $L_n\,(f,x)$ and $L_n^{(k)}(f,x)$ given by

$$L_n\,(f,x) = p_n\,(f,x) + \sum_{j=1}^{n-1}\left[p_j(f,x) - p_{j+1}(f,x)\right]x_j(x)$$

Where $p_j(f) = Q_{m+r-1}\left(f,\mathfrak{I}_j\right)$ is a polynomial of degree $\leq m+r-1$

Now, the polynomial $p_n(f,x)$ in fact, it is a linear operator

$L_p\,[a,b] \to \pi_n$ with $c(p) = c(m,r)]$ such that

$$p_n(f,x) = p_n(f,x) + \sum_{j=1}^{n-1}[p_j(f,x) - p_{j+1}(f,x)]\,T_j(x)$$

Where $T_j(x)$ is defined in lemma 5 with $\xi = m+r$ and $\mu = 7(m+r)$

Satisfies (1) and (2) to justify the above claim we show that

$$J_1 = \left\|\frac{f^{(k)}(x) - L_n^{(k)}(f,x)}{I_j^{rk-k}(x)}\right\|_{p,w} \leq c(p)\omega_{\varphi}^{m+r-rk}(f^{(rk)},n^{-1})_{p,w}$$

And

$$J_2 = \left\|\frac{L_n^{(k)}(f,x) - p_n^{(k)}(f,x)}{I_j^{rk-k}(x)}\right\|_{p,w} \leq c(p)\omega_{\varphi}^{m+r-rk}\left(f^{(rk)},n^{-1}\right)_{p,w}$$

This will prove inequality (1)

We known that $L_n(f,x) = p_j(f,x)\ if\ x \in I_j$, we write for every

$k = 0,\dots,r$ and $k \leq rk \leq r$

$$J_1 \leq c(p)\sum_{j=1}^{n}\int_{I_j}\left|\frac{f^{(k)}(x) - p_j^{(k)}(f,x)}{h_j^{rk-k}}\right|^p dx$$

$$\leq c(p)\sum_{j=1}^{n}\sup_{|h|\leq\delta}\left(\sum_{j=1}^{n}\int_{I_j}\left|\frac{f^{(k)}(x) - p_j^{(k)}(f,x)}{h_j^{rk-k}}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq c(p)\sum_{j=1}^{n}h_j^{(rk-k)p}\ \sup_{|h|\leq\delta}\left(\int_{I_j}\left|f^{(k)}(x) - p_j^{(k)}(f,x)\right|^p w(x)dx\right)^{\frac{1}{p}}$$

$$\leq c(p)\sum_{j=1}^{n} h_j^{(rk-k)p} \sup_{|h|\leq\delta} \left\|f^{(k)} - p_j^{(k)}\right\|_{L_{p,w}}$$

$$\leq c(p)\sum_{j=1}^{n} h_j^{(rk-k)p}\ \omega^{m+r-k}(f^{(rk)}, h_j, I_j)_{p,w}$$

By lemma 2 we get

$$\leq c(p)\ \omega_{\varphi}^{m+r-rk}\left(f^{(rk)}, n^{-1}\right)_{p,w}$$

since $J_2 = \left\|\frac{L_n^{(k)}(f,x)-p_n^{(k)}(f,x)}{I_j^{rk-k}(x)}\right\|_{p,w}$

$$= \sup_{|h|\leq\delta}\int_{I_j} I_j^{(k-rk)p}(x)[\sum_{j=1}^{n-1}\left|p_j^{(k)}(f,x) - p_{j+1}^{(k)}(f,x)\right|\left|X_j(x)\right|$$

$$+\sum_{j=1}^{n-1}\left|p_j^{(k)}(f,x) - p_{j+1}^{(k)}(f,x)\right|\left|T_j(x)\right| + \left|p_n^{(k)}(f,x)\right|)\ w(x)dx]$$

$$= \sup_{|h|\leq\delta}\int_{I_j} I_j^{(k-rk)p}(x)[\sum_{j=1}^{n-1}\left|p_j^{(k)}(f,x) - p_{j+1}^{(k)}(f,x)\right|\left|X_j(x) - T_j(x)\right|$$

$$+\sum_{V=0}^{K-1}\left|p_j^{(V)}(f,x) - p_{j+1}^{(V)}(f,x)\right|\left|T_j(x)^{(K-V)}\right|)\ w(x)dx]$$

let $\left|x_j(x) - T_j(x)\right| = \varphi_j^{\mu-m-r+k+1}$ *and* $\left|T_j^{(k-v)}(x)\right| = h_j^{v-k}\varphi_j^{\mu-m-r+k+1}$

$$\leq c(p)\sup_{|h|\leq\delta}\int_{I_j} I_j^{(k-rk)p}(x)[\sum_{j=1}^{n-1}(\left\|p_j^{(k)}(f) - p_{j+1}^{(k)}(f)\right\|_{p,w}\ \varphi_j^{\mu-m-r+k+1}$$

$$+\sum_{v=0}^{k-1}\left\|p_j^{(v)}(f) - p_{j+1}^{(v)}(f)\right\|_{p,w} h_j^{v-k}\ \varphi_j^{\mu-m-r+k+1})^{\frac{1}{p}} w(x)dx$$

By lemma 6 we used $\sum_{j=1}^{n}\ \varphi_j^{\alpha} \leq c, \alpha \geq 2$. So

$$J_2 \leq c(p)\int_{I_j} I_j^{(k-rk)p}(x)\ [\ \sum_{j=1}^{n-1}[\left\|p_j(f) - p_{j+1}(f)\right\|_{p,w}\ h_j^{-1}\varphi_j^{\frac{\mu}{2}}\ ]w(x)dx$$

$$\leq c(p)\int_{I_j} I_j^{(k-rk)p}(x)\sum_{j=1}^{n-1}\ \left\|p_j(f) - p_{j+1}(f)\right\|^p\ h_j^{-kp}\varphi_j^{\frac{\mu}{2-r}}\varphi_j^{rp} w(x)dx$$

$$\leq c(p)\sum_{j=1}^{n-1}\ \left\|p_j(f) - p_{j+1}(f)\right\|^p\ h_j^{-rkp} w(x)dx\ \int_{I_j}\varphi_j^{\frac{\mu}{2-r}}\ dx.$$

since $h_j\varphi_j \leq I_j$ and $\varphi_j \leq 1$ *and* $I_j^{(k-rk)p}\varphi_j^{rp} \leq c(p)h_j^{(k-rk)p}\varphi_j^{(r+k-rk)p} \leq c(p)h_j^{(k-rk)p}$

And since

$\int_{I_j}\varphi_j^{\alpha}\ dx\ \leq c(\alpha)h_j\ \forall\ \alpha \geq 2$. So

$$J_2 \leq c(p)\sum_{j-1}^{n-1} h_j^{1-rkp}\left\|p_j(f) - p_{j+1}(f)\right\|_{p,w}$$

$$J_2 \leq c(p)\sum_{j-1}^{n-1} h_j^{-rkp}\int_{I_j}\left|p_j(f,x) - p_{j+1}(f,x)\right|^p\ w(x)dx$$

$$J_2\ \leq c(p)\sum_{\mathrm{j-1}}^{\mathrm{n-1}} h_j^{-rkp}\int_{\mathrm{I_j}}\left|\mathrm{f(x) - p_j(f,x)}\right|^p w(x)dx$$

By using(lemma 2) and(lemma 4) we have

$$J_2 \leq c(p)\sum_{j=1}^{n-1} h_j^{-rkp}\omega^{m+r}(f, h_j, I_j)_{p,w} \leq c(p)\sum_{j=1}^{n-1}\omega^{m+r-rk}(f^{(rk)}, h_j, I_j)_{p,w}$$

$\leq c(p)\omega_{\boldsymbol{\varphi}}^{m+r-rk}(f^{(rk)},n^{-1})_{p,w}$

So the proof (1) is complete.

In the same way for any $r^{\sim}, 0 \leq r^{\sim} \leq r$

And $k = m + r$ we write, we can estimate (2).

$J_3 = \left\|I_j(x)^{k-r^{\sim}} p_n^{(k)}(f,x)\right\|_{p,w}$

$\leq \int_{-1}^{1} I_j(x)^{(k-r^{\sim})p} [\sum_{j=1}^{n-1} \sum_{v=0}^{m+r-1} \binom{k}{v} \left|p_j^{(v)}(f,x) - p_{j+1}^{(v)}(f,x)\right|\left|T_j^{(k-v)}(x)\right|]^p \, w(x)dx$

$\leq c(p) \int_{-1}^{1} I_j^{(k-r^{\sim})p}(x) \left[\sum_{j=1}^{n-1} \left\|p_j(f) - p_{j+1}(f)\right\|_{p,w} h_j^{-k} \varphi_j^{\mu/2}\right]^p dx$

$J_3 \leq c(p)\omega_{\boldsymbol{\varphi}}^{m+r-r^{\sim}}(f^{(r^{\sim})},n^{-1})_{p,w}$

Then$\left\|I_j(x)^{k-r^{\sim}} p_n^{(k)}(f,x)\right\|_{p,w} \leq c(p)\omega_{\boldsymbol{\varphi}}^{m+r-r^{\sim}}\left(f^{(r^{\sim})},n^{-1}\right)_{p,w}$

Then the proof of theorem 1 is done.

Conclusion

In conclusion, after we knew the norm, modulus and doubling weight Ditizion Totik modulus of smoothness, we were able to find features and properties of the modulus For example in (lemma 2) we explained that

$$\sum_{j=1}^{n} \omega_{\varphi}^{r}\left(f,h_j,I_j\right)_{p,w} \leq c(p)\omega_{\varphi}^{r}\left(f,h_j^{-1}\right)_{p,w}$$

Either in (lemma 5) we have explained the relationship between the function norm and the polynomial which helped us to proof (theorem 2)which relates the norm to (function with polynomial) and modulus like as

$$\|f - p_{m-1}(f)\|_{L_{p,w}} \leq c(p)\omega_m^r(f,b-a)_{p,w}$$

Finally, we were able to proof a special case for the modulus as shown in (lemma 9), such that the function is equal to polynomial $(f = Q)$ for each set whose measure is not equal to zero. And our research has several uses, the most important of which is image processing.

References

[1] A.F. TIMAN(1951): A strengthening of Jackson's theorem on the best approximation of continuous functions on a finite segment of the real axis. Dokl. Akad. Nauk USSR, 78:17-20 (in Russian).

[2] A.F. TIMAN(1951): A strengthening of Jackson's theorem on the best approximation of continuous functions on a finite segment of the real axis. Dokl. Akad. Nauk USSR, 78:17-20 (in Russian).

[3] R, M. TRIGUB(1962): Approximation of functions by polynomials with integral coefficients. Izv. Akad. Nauk SSSR, Ser. Mat., 26:261-280 (in Russian).

[4] YU. A. BRUDNYI(1963): Generalization of a theorem ofA. F. Timan. Soviet Math. Dokl., 4:244-247

[5] S. A. TELYAKOVSKll(1966): Two theorems on the approximation of functions by algebraic polynomials. Mat. Sb., 70:252-265 (in Russian) (English translation (1966): American Mathematical Society Translations, Series 2, 77:163-178).

[6] I.E. GOPENGAUZ(1967): A question concerning the approximation of functions on a segment and a region with corners, Theor. Funkcii Funkcional. Anal. Pril., 4:204-210 (in Russian

[7] R.A. DEVORE (1977): Pointwise approximation by polynomials and splines. In: Theory of Approximation of Functions (S. B. Stechkin and S. A. Telyakovskii, eds.). Proc. Intern. Conf., Kaluga, 1975, Nauka, Moskva, pp. 132-141

[8] E. HINNEMANN,H. H. GONSKA (1983): Generalization of a theorem of DeVore. In: Approximation

Theory IV (C. K. Chui, L. L. Schumaker, J. W. Ward, eds.). New York: Academic Press, pp. 527-532

[9] H. GONSKA,E. HINNEMANN(1985): Pointwise estimations of approximations by algebraic polynomials. Acta Math. Hungar., 46(3-4):243-254 (in German)

[10] W. LI (1986): On Timan type theorems in algebraic polynomial approximation. Acta Math. Sinica, 29(4):544-549 (in Chinese).

[11] R. DAHLHAUS(1989): Pointwise approximation by algebraic polynomials. J. Approx. Theory, 57:274-277.

[12] I.A.Shevchuk, "Approximation by polynomials and Traces of the Functions Continuous on an Interval," in kiev:Naukova Dumka 1992, (in Russian).

[13] Kopotun, Kirill. "A NOTE ON SIMULTANEOUS APPROXIMATION IN Lp [—1, 1](1≤< ρ<∞)." Analysis 15.2 (1995): 151-158.

[14] Petrushev, Penco Petrov, and Vasil Atanasov Popov. Rational approximation of real functions. No. 28. Cambridge University Press, 2011.