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Attraction Properties and Asymptotically Equivalent of Random Dynamical System

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Abstract. The main aim of this paper is study the concept of asymptotically equivalent of random dynamical system, in the time space (which is consider as a locally compact group with Harr measure). The concept of asymptotically equivalent of random dynamical system is defined and some essential properties are proved. Further the relation between the asymptotically equivalent and omega limit set on $\mathbb{R}$, stable and eventually stable are given.

Key wards: random dynamical system, asymptotically equivalent, omega limit set, attraction, weak attraction,

1. Introduction

One of the furthermost imperative detections in mathematical physics through the past 50 years is that of finite-dimensional attractors in mathematical models for fluid dynamics. Nevertheless, all the analysis breaks down the instant one desires to yield random effects on the system under inquiry in justification. Especially, while imperiling the system to additive white noise, there is no chance that bounded subsets of the state space stay invariant. White noise impulsions the system out of every bounded set with probability one.

Attractors for deterministic systems are moderately fine studied. Temam [8] provides a comprehensive presentation. In the deterministic case altered notions of attractor have been hosted. The differences between them mainly concern speed of convergence to the attractor. This amounts essentially to the question whether certain points of the phase space are elements of the attractor. For stochastic systems a greater variety of definitions is possible. Two of them, completely different from the one of this paper, have been introduced previously. Brzezniak, Capinski and Flandoli [5], consider the O-limit set for $t \longrightarrow +\infty$ of the trajectories. Morimoto [6] and Schmalfug [7]f are concerned with attractors for the Markov semigroup generated by a stochastic differential equation. In this paper we think of the attractor as a subset of the phase space (as in [5]), instead of the space of probability measures. However, we define the attractor as the Omega-limit set at time $t = 0$ of the trajectories "starting in bounded sets at time $t \longrightarrow +\infty$ (roughly speaking). Equivalently, we detect a random subset of the phase space which moves accordingly to the dynamics in a stationary manner, attracting all trajectories starting from deterministic or random initial conditions. While the definition of [5*] is of interests for systems with real noise, the notion developed here is useful for the white noise case as well.The notions of stabilities and attraction properties such as the attractors and the region of attraction are rather important to determine the behaviors of the deterministic dynamical systems defined on a metric space [1],**[2].**

In [3] Shigeo KONO investigate the problem that to what extent the stability properties and the attraction properties are preserved through the asymptotic equivalence.

In [9] Summary. A criterion for existence of global random attractors for RDS is established. Existence of invariant Markov measures supported by the random attractor is proved. For SPDE this yields invariant measures for the associated Markov semigroup. The results are applied to reaction diffusion equations with additive white noise and to Navier-Stokes equations with multiplicative and with additive white noise.

In [11],[12] Mohammed Shanan Imran and Ihsan Jabbar Kadhim they introduce Non-autonomous invariant sets and attractors Random Dynamical System and Stability and Asymptotic stability of Closed Random Sets respectively.

2. Attractors of Random Dynamical System

The aim of this section is to study region of attraction, region of weak attraction. Also some types of stability of closed random sets are defind and studied.

Definition(2.1.1)[4]: The 5-tuple $(\mathbb{R}, \Omega, F, P, \theta)$ is called a ***metric dynamical system*** (Shortly MDS) if (Ω, F, P) is a probability space and

(i) $\theta: \mathbb{R} \times \Omega \to \Omega$ is $(B(\mathbb{R}) \otimes F, F)$ $-$measurable,

(ii) $\theta(0, \omega) = Id_{\Omega}$,

(iii) $\theta(t+s,\omega)=\theta(t,\theta(s,\omega))$ and

(iv) $P(\theta(t)F)=P(F)$, for every $F\in F$ and every $t\in\mathbb{R}$.

Definition (2.1.2) [4]: A topological random dynamical system on the topological space X over (or covering, or extending) an MDS $(\mathbb{R},\Omega,F,P,\theta)$ with time is a mapping $\varphi:\mathbb{R}\times\Omega\times X\to X$, with the following properties:

(i) φ is continuous

(ii) Cocycle property: The mappings $\varphi(t,\omega)\coloneqq\varphi(t,\omega,\cdot):X\to X$ form a cocycle over $\theta(\cdot)$, i. e. they satisfy

$\varphi(0,\omega)=id_X$ for all $\omega\in\Omega$,

$\varphi(t+s,\omega)=\varphi(t,\theta(s)\omega)\circ\varphi(s,\omega)$ for all $s,t\in\mathbb{R},\omega\in\Omega$.

Definition (2.1.3) [4]: Let (Ω,F) be a measurable space and (X,d) be a metric space which is considered a measurable space with Borel $\sigma-$ algebra $B(X)$. The set-valued function $M:\Omega\to B(X),\omega\longmapsto M(\omega)$, is said to be ***random set*** if for each $x\in X$ the function $\omega\longmapsto d(x,M(\omega))$ is measurable. If $M(\omega)$ is connected closed (compact) for all $\omega\in\Omega$, it is called a ***random connected closed(compact) set***.

Proposition (2.1.4) [4]: Let the set-valued function $M:\Omega\to B(X)$ take values in the subspace of closed subsets of a Polish space (complete, separable, second countable) X. Then:

(i) M is a random closed set if and only if for all open sets $U\subset X$ the set $\{\omega: M_\omega\cap U\neq\emptyset\}$ is measurable.

(ii) If M is a random closed set then $graph(M)\in F\otimes B$.

Definition (2.1.5)[4]: Let $M:\omega\longmapsto M(\omega)$ be a multifunction. We call the multifunction

$$\omega\longmapsto\gamma_M^t(\omega)\coloneqq\bigcup_{\tau\geq t}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)$$

the tail (from the moment t) of the pull back trajectories emanating from M.

Definition (2.1.6) [4]: Let (θ,φ) be a measurable RDS and $M\subset\Omega\times X$ a set.

(i) M is called forward invariant if for $t>0$

$$M(\omega)\subset\varphi(t,\omega)^{-1}M(\theta(t,\omega))\quad P\text{-a.s.}$$

equivalently

$$\varphi(t,\omega)M(\omega)\subset M(\theta(t,\omega))\ P\text{-a.s.}$$

(ii) M is called invariant if for all $t\in P$

$$M(\omega)=\varphi(t,\omega)^{-1}M(\theta(t,\omega))\ \ P\text{-a.s a.s.},$$

for two-sided time equivalent to

$$\varphi(t,\omega)M(\omega)=M(\theta(t,\omega))\quad P\text{-a.s a.s.}$$

Definition (2.1.7)[4]: Let $M:\omega\longmapsto M(\omega)$ be a multifunction. We call the multifunction

$$\Gamma_M(\omega)=\bigcap_{t>0}\overline{\gamma_M^t(\omega)}=\bigcap_{t>0}\overline{\bigcup_{\tau\geq t}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$$

the (pull back) omega-limit set of the trajectories emanating from M

Theorem (2.1.8)[4]: For every random set M in, where

$$\Gamma_M^+(\omega)=\{y\in X:\exists\ a\ seq.\{t_n\}in\ R^+,t_n\longrightarrow+\infty\ and\ \{x_n\}\in M(\theta_{-t_n}\omega)with\ \varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow y\}$$

Definition(2.1.9)[4]: (universe of sets) Let $\mathfrak{D}$ be a family of random closed sets, if $\{D_2(\omega)\}$ possesses the property $D_2(\omega)\subset D_1(\omega)$ for all $\omega\in\Omega$, then $D_2\in\mathfrak{D}$. Then $\mathfrak{D}$ is called the universe of sets.

Definition(2.1.10):[4] Let $\mathfrak{D}$ be a universe. A random closed set A(ω) from $\mathfrak{D}$ is said to

attracting in $\mathfrak{D}$ if

$$lim_{t\to+\infty}d_X\{\psi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\mid A(\omega)\}=0,\omega\in\Omega\ ,for\ all\ D\in\mathfrak{D}.$$

Proposition(2.1.11): Let $\mathfrak{D}$ be a universe .A random closed set A(ω) from $\mathfrak{D}$ is an **attractor** of the *RDS* (θ,ψ) in $\mathfrak{D}$ if and only if

$\Gamma_{D,\psi}(\omega) \neq \emptyset$ and $\Gamma_{D,\psi}(\omega) \subset A(\omega)$, **for all** $D \in \mathfrak{D}$.

Definition(2.1.12): A random closed set is said to be a weak attractor of the *RDS* (θ, ψ) if there is a tempered random variable δ a sequence $\{t_n\}$ in R^+, $t_n \longrightarrow +\infty$, such that

$$d\big(\psi(t_n, \theta_{-t_n}\omega)y, M(\omega)\big) \longrightarrow 0, \text{ for every } y \in S(M, \delta(\omega)) \text{ and } \omega \in \Omega.$$

Definition(2.1.13)[10]: Let M be a random set compact set in a locally compact space X .

(i) The set

$$A^*_{M,\psi}(\omega) := \big\{x \in X: \exists\, \{t_n\} in\ R^+ \ \ni\ t_n \longrightarrow +\infty\ and\ d(\psi(t_n, \theta_{-t_n}\omega)x, M(\omega)) \longrightarrow 0\,\big\}$$

is called the **region of weak attraction** of the set M.

(ii) The set

$$A_{M,\psi}(\omega) := \{x \in X:\ d(\psi(t, \theta_{-t}\omega)x, M(\omega)) \longrightarrow 0\ as\ \ t \longrightarrow +\infty\,\}$$

is called the **region of attraction** of the set M.

Definition(2.1.14): Let (θ, ψ) be a random dynamical system. A closed random set M in X is said to be **positive stable** if for every tempered random variable $\varepsilon: \Omega \longrightarrow R^+$ and $x \in M$, there is a tempered random variable $\delta \equiv \delta_{x,\varepsilon}: \Omega \longrightarrow R^+$ such that

$\gamma^t_S(\omega) \subseteq S(M, \varepsilon)$**, where** $S(\omega) \equiv S(x, \delta(\omega))$

Definition(2.1.15): A closed random set M in a locally compact space X is said to be **eventually stable** with respect to (θ, ψ) if for every tempered random variable ε, there exists $\tau > 0$ and tempered random variable δ such that

$$\psi(t, \theta_{-t}\omega)S_\delta(\theta_{-t}\omega) \subset S_\varepsilon(\omega), \text{ for every } t \geq \tau,$$

where $S_\varepsilon(\omega) \equiv S(M, \varepsilon)$ and $S_\delta(\omega) \equiv S(M, \delta)$.

Definition(2.1.16): A closed random set M in a locally compact space X is said to be **asymptotically stable** with respect to (θ, ψ) if M is an attractor of (θ, ψ) and stable with respect to (θ, ψ).

3. Asymptotically Equivalent in Random Dynamical Systems

In this section the concepts of asymptotically equivalent in random dynamical systems is defined and some essential properties are investigated. As the main property we characterize the asymptotically equivalent in terms of omega limit sets.

Definition(3.1.1): We say a dynamical system (θ, ψ_1) is asymptotically equivalent to (θ, ψ_2) in the universal $\mathfrak{D}$ if for all $D_1, D_2 \in \mathfrak{D}$, the following hold:

$$d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y) \longrightarrow 0 \quad \text{as } (t \longrightarrow +\infty)$$

is valid, for every $x \in D_1(\theta_{-t}\omega)$ and $y \in D_2(\theta_{-t}\omega)$, and denote this fact as follows $(\theta, \psi_1) \sim (\theta, \psi_2)$ or $\psi_1 \sim \psi_2$

Proposition(3.1.2) The asymptotic equivalence is symmetric as well as transitive relation.

Proof. The symmetry follows from the fact that

$$d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y) = d(\psi_2(t, \theta_{-t}\omega)y, \psi_1(t, \theta_{-t}\omega)x).$$

The transitivity follows from the triangle inequality :

$$d(\psi_1(t, \theta_{-t}\omega)x, \psi_3(t, \theta_{-t}\omega)z) \leq d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y) + d(\psi_2(t, \theta_{-t}\omega)y, \psi_3(t, \theta_{-t}\omega)z)$$

Corollary(3.1.3): The asymptotic equivalence of the dynamical systems on a singleton is an equivalence relation.

Theorem(3.1.4): Let $\mathfrak{D}$ be a universal and (θ, ψ) be an RDS. If $\Gamma_{D,\psi}$ is nonempty compact, then

$$d(\psi(t, \theta_{-t}\omega)x, \Gamma_{D,\psi}(\omega)) \ = 0 \qquad (1)$$

for every $D \in \mathfrak{D}$ and every $x \in D(\theta_{-t}\omega)$.

Proof Assume contrary that (1) does not hold, then there is a sequence $\{t_n\}$ in R^+ with $t_n \longrightarrow +\infty$ and a $\beta > 0$ such that

$$d\left(\psi\left(t_n, \theta_{-t_n}\omega\right)x_n, \Gamma_{D,\psi}(\omega)\right) \geq \beta > 0 \qquad (2)$$

for some $D \in \mathfrak{D}$ and $x_n \in D(\theta_{-t_n}\omega)$

Since

$$d\left(\psi\left(t_n, \theta_{-t_n}\omega\right)x_n, \Gamma_{D,\psi}(\omega)\right) = d(\psi\left(t_n, \theta_{-t_n}\omega\right)x_n, \gamma_D^k(\omega)) \text{, for every } \omega \in \Omega,$$

then

$$d(\psi\left(t_n, \theta_{-t_n}\omega\right)x_n, \gamma_D^k(\omega)) \geq \beta > 0 \qquad (3)$$

Now, the sequence $\{\psi\left(t_n, \theta_{-t_n}\omega\right)x_n\} : \psi\left(t_n, \theta_{-t_n}\omega\right)x_n \in \overline{\gamma_D^k(\omega)}$ such that

$$\psi\left(t_n, \theta_{-t_n}\omega\right)x_n = y \in \overline{\gamma_D^k(\omega)},$$

since $\overline{\gamma_D^k(\omega)}$is compact. On the other hand from (2) it follows that

$$d\left(y, \Gamma_{D,\psi}(\omega)\right) \geq \beta > 0 \text{ .}$$

This is a contradiction. Thus we must have

$$d(\psi(t, \theta_{-t}\omega)x, \Gamma_{D,\psi}(\omega)) = 0.$$

Theorem(3.1.5): Let (θ, ψ_1) and (θ, ψ_2) be two RDS's on the locally compact metric space X and let $\mathfrak{D}$ be a universal such that Γ_{D_1,ψ_1} and Γ_{D_2,ψ_2} are both non-empty and compact random for any $D_1, D_2 \in \mathfrak{D}$. Then

$(\theta, \psi_1) \sim (\theta, \psi_2)$ in $\mathfrak{D}$ if and only if $\Gamma_{D_1,\psi_1} \cap \Gamma_{D_2,\psi_2} \neq \emptyset$, for all $D_1, D_2 \in \mathfrak{D}$.

Proof Suppose that $(\theta, \psi_1) \sim (\theta, \psi_2)$ in $\mathfrak{D}$, then

$$d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y) \longrightarrow 0 \quad \text{as } (t \longrightarrow +\infty).$$

for every $x \in D_1(\theta_{-t}\omega)$ and $y \in D_2(\theta_{-t}\omega)$. The sets Γ_{D_1,ψ_1} and Γ_{D_2,ψ_2} are both non-empty and compact random sets in a locally compact spaces X, then by Theorem(3.1.4)

$$d(\Gamma_{D_1,\psi_1}(\omega), \psi_1(t, \theta_{-t}\omega)x) = 0 \qquad (1)$$

for every $D_1 \in \mathfrak{D}$ and every $x \in D_1(\theta_{-t}\omega)$, and

$$d(\psi_2(t, \theta_{-t}\omega)y, \Gamma_{D_2,\psi_2}(\omega)) = 0 \qquad (2)$$

for every $D_2 \in \mathfrak{D}$ and every $y \in D_2(\theta_{-t}\omega)$

Using (1) and (2) in the inequality

$$d\left(\Gamma_{D_1,\psi_1}(\omega), \Gamma_{D_2,\psi_2}(\omega)\right) \leq d\left(\Gamma_{D_1,\psi_1}(\omega), \psi_1(t, \theta_{-t}\omega)x\right) + d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y)$$
$$+d(\psi_2(t, \theta_{-t}\omega)y, \Gamma_{D_2,\psi_2}(\omega)),$$

Then

$$d\left(\Gamma_{D_1,\psi_1}(\omega), \Gamma_{D_2,\psi_2}(\omega)\right) = 0 \text{ for every } D_1, D_2 \in \mathfrak{D} \text{ .} \qquad (3)$$

Since $\Gamma_{D_1,\psi_1}(\omega)$ and $\Gamma_{D_2,\psi_2}(\omega)$ are closed for every $D_1, D_2 \in \mathfrak{D}$, , then

$$\Gamma_{D_1,\psi_1} \cap \Gamma_{D_2,\psi_2} \neq \emptyset, \text{ for all } D_1, D_2 \in \mathfrak{D}. \qquad (4)$$

Conversely, the assumption (4) implies (3). Using (1),(2) and (3) to the inequality

$$d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y) \leq d\left(\psi_1(t, \theta_{-t}\omega)x, \Gamma_{D_1,\psi_1}(\omega)\right)$$
$$+d\left(\Gamma_{D_1,\psi_1}(\omega), \Gamma_{D_2,\psi_2}(\omega)\right) + d(\psi_2(t, \theta_{-t}\omega)y, \Gamma_{D_2,\psi_2}(\omega))$$

for every $x \in D_1(\theta_{-t}\omega)$ and $y \in D_2(\theta_{-t}\omega)$. Thus we have

$$d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y) \longrightarrow 0 \quad \text{as } (t \longrightarrow +\infty)$$

for every $x \in D_1(\theta_{-t}\omega)$ and $y \in D_2(\theta_{-t}\omega)$.i.e., $(\theta, \psi_1) \sim (\theta, \psi_2)$.

Corollary (3.1.6) Let (θ, ψ_1) and (θ, ψ_2) be two RDS's on the compact metric space X and let $\mathfrak{D}$ be a universal such that Γ_{D_1,ψ_1} and Γ_{D_2,ψ_2} are both non-empty and compact random for any $D_1, D_2 \in \mathfrak{D}$. Then

$$(\theta, \psi_1) \sim (\theta, \psi_2) \text{ in } \mathfrak{D} \text{ if and only if } \Gamma_{D_1,\psi_1} \cap \Gamma_{D_2,\psi_2} \neq \emptyset, \text{ for all } D_1, D_2 \in \mathfrak{D}.$$

Proof: This follows from the fact that the compact space is locally compact.

Theorem(3.1.7): Let X be a compact metric space and $(\theta,\psi_1)\sim(\theta,\psi_2)$ on a non-empty random set D. If $M\subset D$ is an attractor of (θ,ψ_1), then M is a weak attractor of (θ,ψ_2).

Proof. Since $M\subset D$ is an attractor of (θ,ψ_1), then there exists a tempered random variable $\varepsilon>0$ such that

$$\Gamma_{x,\psi_1}(\omega)\neq\emptyset \text{ and } \Gamma_{x,\psi_1}(\omega)\subset M(\omega), \text{ for every } x\in S(M,\varepsilon).$$

On the other hand for every point $x\in M$ there exists a neighborhood $U(x)$ such that $U(x)\subset D$. We take a Lebesgue number λ of the covering

$$\{U(x): x\in M, U(x)\subset D(\omega)\}.$$

Then

$$S(M,\lambda)=\cup_{x\in M}S(x,\lambda)\subset\cup_{x\in M}U(x)\subset D(\omega).$$

We can choose ε to satisfy $\varepsilon\leq\lambda$. Thus

$$S(M,\varepsilon)\subset D(\omega). \qquad (1)$$

Then

$$\Gamma_{p,\psi_1}(\omega)\cap\Gamma_{q,\psi_2}(\omega)\neq\emptyset \text{ for every } p\in S(M,\varepsilon) \text{ and } q\in D(\omega),$$

because of (1), the assumption $(\theta,\psi_1)\sim(\theta,\psi_2)$ and Corollary (3.1.6). On the other hand

$$\Gamma_{p,\psi_1}(\omega)\neq\emptyset \; and \; \Gamma_{p,\psi_1}(\omega)\subset M(\omega) \text{ for every } p\in S(M,\varepsilon).$$

Hence

$$\Gamma_{q,\psi_2}(\omega)\cap M(\omega)\neq\emptyset, \text{ for every } p\in S(M,\varepsilon).$$

This shows that M is a weak attractor of (θ,ψ_2).

Theorem(3.1.8): Let X be a compact metric space and $(\theta,\psi_1)\sim(\theta,\psi_2)$ on a non-empty random open set D. If a compact random set $M\subset D$ is an stable with respect to (θ,ψ_1), then M is a eventually stable with respect to (θ,ψ_2).

Proof Since D is an open neighborhood of M then there exists a $\delta'>0$ such that

$$S(M,\delta')\subset D \qquad (1)$$

Note that a positive number δ' can be considered as a constant random variable and hence a tempered random variable. Because of M is stable with respect to (θ,ψ_1), then for every tempered random variable ε, there exist $\tau>0$ and a tempered random variable $\delta''>0$ such that

$\psi_1(t,\theta_{-t}\omega)S_{\delta''}(\theta_{-t}\omega)\subset S_{\frac{\varepsilon}{2}}(\omega)$, for every $t\geq\tau$,

where $S_{\frac{\varepsilon}{2}}(\omega)\equiv S(M,\frac{\varepsilon}{2})$ and $S_{\delta''}(\omega)\equiv S(M,\delta'')$. Set

$$\delta(\omega):=\{\delta'(\omega),\delta''(\omega)\},$$

where $\delta'(\omega)\equiv\delta'$, for every $\omega\in\Omega$. Thus for every tempered random variable ε, there exists a tempered random variable $\delta''>0$ such that

(i) $\psi_1(t,\theta_{-t}\omega)S_\delta(\theta_{-t}\omega)\subset S_{\frac{\varepsilon}{2}}(\omega)$, for every $t\in R^+$, and

(ii) $S_\delta(\omega)\subset D$.

Since $D\in\mathfrak{D}$ and $S_\delta(\omega)\subset D$, then $S_\delta(\omega)\in\mathfrak{D}$. By hypothesis $(\theta,\psi_1)\sim(\theta,\psi_2)$ in the universal $\mathfrak{D}$ then we have

$$d(\psi_1(t,\theta_{-t}\omega)x,\psi_2(t,\theta_{-t}\omega)y)<\varepsilon(\omega)/2$$

is valid, for every $x\in S_\delta(\theta_{-t}\omega)$ and $y\in D(\theta_{-t}\omega)$.

Using these results to the inequality

$$d(\psi_2(t,\theta_{-t}\omega)y,M)\leq d(\psi_2(t,\theta_{-t}\omega)y,\psi_1(t,\theta_{-t}\omega)x)+d(\psi_1(t,\theta_{-t}\omega)x,M),$$

we get for every tempered random variable ε, there exists $\tau>0$ and tempered random variable δ such that

$d(\psi_2(t,\theta_{-t}\omega)y,M)<\frac{\varepsilon(\omega)}{2}+\frac{\varepsilon(\omega)}{2}=\varepsilon(\omega)$, for every $t\geq\tau$, and $y\in S_\delta(\theta_{-t}\omega)$.

Consequently, M is eventually stable with respect to (θ,ψ_2).

Theorem(3.1.9): Let X be a compact metric space and $(\theta,\psi_1)\sim(\theta,\psi_2)$ on a non-empty random open set D. If a compact random set $M\subset D$ is an asymptotically stable with respect to (θ,ψ_1), then M is a asymptotically stable with respect to (θ,ψ_2).

Proof: Use the same technique that given in the proof of Theorem (3.1.8)

Theorem(3.1.10): Let X be a compact metric space and $(\theta, \psi_1) \sim (\theta, \psi_2)$ on a non-empty random open set D. If M a non-empty compact random set in X , then

(i) $A^*_{M,\psi_1}(\omega) \cap D \neq \emptyset$, then $D \subset A^*_{M,\psi_2}(\omega)$,

(ii) $A_{M,\psi_1}(\omega) \cap D \neq \emptyset$, then $D \subset A_{M,\psi_2}(\omega)$

Proof. (i) Suppose that $A^*_{M,\psi_1}(\omega) \cap D \neq \emptyset$, then there exists $x \in A^*_{M,\psi}(\omega) \cap D$. So there exists a sequence $\{t_n\}$ such that

(1) $t_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$,

(2) $d(\psi_1(t_n, \theta_{-t_n}\omega)x, M(\omega)) \longrightarrow 0$ as $n \longrightarrow +\infty$,

(3) $d(\psi_1(t_n, \theta_{-t_n}\omega)x, \psi_2(t_n, \theta_{-t_n}\omega)y) \longrightarrow 0$ as $n \longrightarrow +\infty$.

On the other hand , for every $y \in D$, we have

$$d(\psi_2(t_n, \theta_{-t_n}\omega)y, M(\omega)) \leq d(\psi_2(t_n, \theta_{-t_n}\omega)y, \psi_1(t_n, \theta_{-t_n}\omega)x) + d(\psi_1(t_n, \theta_{-t_n}\omega)x, M(\omega)),$$

which implies that

$$d(\psi_2(t_n, \theta_{-t_n}\omega)y, M(\omega)) \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

So by definition of $A^*_{M,\psi_2}(\omega)$ we have $y \in A^*_{M,\psi_2}(\omega)$. Consequently $D \subset A^*_{M,\psi_2}(\omega)$.

(ii) Suppose that $A_{M,\psi_1}(\omega) \cap D \neq \emptyset$, then there exists $x \in A_{M,\psi_1}(\omega) \cap D$. Then

$d(\psi_1(t, \theta_{-t}\omega)x, M(\omega)) \longrightarrow 0$ as $t \longrightarrow +\infty$,

and

$d(\psi_1(t, \theta_{-t}\omega)x, \psi_2(t, \theta_{-t}\omega)y) \longrightarrow 0$ as $t \longrightarrow +\infty$ for every $y \in D$

are valid. Applying these results to the inequality

$$d(\psi_2(t, \theta_{-t}\omega)y, M(\omega)) \leq d(\psi_2(t, \theta_{-t}\omega)y, \psi_1(t, \theta_{-t}\omega)x) + d(\psi_1(t, \theta_{-t}\omega)x, M(\omega)),$$

Then

$$d(\psi_2(t, \theta_{-t}\omega)y, M(\omega)) \longrightarrow 0 \text{ as } t \longrightarrow +\infty \text{ for every } y \in D.$$

So by definition of $A_{M,\psi_2}(\omega)$ we have $y \in A_{M,\psi_2}(\omega)$. Consequently $D \subset A^*_{M,\psi_2}(\omega)$.

Corollary(3.1.11): Under the similar assumption theorem (3.1.10) the following are hold:

(i) $A^*_{M,\psi}(\omega) \cap D \neq \emptyset$, then $D \subset A^*_{M,\psi}(\omega)$,

(ii) $A_{M,\psi}(\omega) \cap D \neq \emptyset$, then $D \subset A_{M,\psi}(\omega)$.

Proof. This follows from the fact that $(\theta, \psi_1) \sim (\theta, \psi_1)$ and the above theorem.

Corollary(3.1.12): Under the similar assumption theorem (3.1.10) the following are hold:

(i) $A^*_{M,\psi_1}(\omega) \subset D$, then $A^*_{M,\psi_1}(\omega) \subset A^*_{M,\psi_2}(\omega)$,

(ii) $A_{M,\psi_1}(\omega) \subset D$, then $A_{M,\psi_1}(\omega) \subset A_{M,\psi_1}(\omega)$.

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