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Third Order Qualitative Inspection of Nonlinear Neutral Difference Equation

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Abstract - This paper aim is to find out the nonlinear neutral third order difference equation in the form We obtain some criteria to ensure that every solution of this equation are either oscillatory or converges to zero. These results improved some of the existing results. We derived this using summation averaging technique and comparison principle.

Index Terms - Oscillation; Nonlinear; Neutral Difference Equation. AMS subject classification: 39A10.

INTRODUCTION

In the present paper, we have obtained some criteria for nonlinear neutral third order difference equation

$$\Delta\left(a_e \Delta^2\left(w_e \pm p_e w_{e-l}\right)\right) - q_e f\left(w_{e-m}\right) = 0, e \in \mathrm{N}_0 \qquad (1)$$

Where the following conditions are assumed to be hold

(H_1) $\{a_e\}, \{p_e\}, \{q_e\}$ are positive real sequence where $\mathrm{N}_0 = \{e_0, e_0+1, e_0+2, \ldots\}$ and e_0 is a nonnegative integer.

(H_2) $\sum_{e=e_0}^{\infty} \frac{1}{a_e} = \infty$

(H_3) $\gamma \le p_e \le 1$ for $\gamma \in (0,1)$

(H_4) l, m are positive integer and f is a continuous real valued function.

(H_5) $f : \mathrm{N}_0 \times R \to [0,\infty)$ and $\{t_e\}$ be the nonnegative real sequence, where $u \ne 0$ and $A > 0$ such that $\frac{f(u)}{u} \ge At_e$.

Equation (1) satisfy the real sequence $\{w_e\}$ for all $e \in \mathrm{N}_0$ and $\sup\{|w_e| : e \ge \mathrm{N}\} > 0$ for all $e \in \mathrm{N}_0$. The solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative and it is nonoscillatory otherwise.

The problem of determining the oscillation criteria for neutral difference equation have been receiving great attention in the last few decades since these type of equation arise in the study of economics, mathematical biology, and many other areas of mathematics [7,9].

In the last few decade, lots of research done on oscillation of third order difference equations, e.g. ([12] [11], [10], [8], [6], [5], [4], [3], [2], [1]) and the references cited therein.

In [2], the authors obtained some oscillatory properties of quasilinear neutral third order difference equation in the form $\Delta\left(a_n\left(\Delta^2\left(x_n + b_n x_{(n-\delta)}\right)\right)^{\alpha}\right) + q_n x^{\alpha}{}_{n+1-\tau} = 0, n \ge 0$ by using Riccati transformation.

In [3], the authors considered the condition for oscillation of difference equation for third order in the form $\Delta\left(a_n\left(\Delta^2\left(x_n \pm b_n h x_{(n-\delta)}\right)\right)^{\alpha}\right) + q_n f\left(x_{n+1-\tau}\right) = 0$ are established.

In [4], the authors considered the oscillation of third order nonlinear neutral difference equation of the form $\Delta\left(a_n\left(\Delta^2\left(x_n \pm b_n x_{(n-\delta)}\right)\right)^{\alpha}\right) + q_n x^{\alpha}{}_{n+1-\tau} = 0$ are established.

In [6], the authors considered the third order nonlinear difference equation of the form

$\Delta\left(c_n\Delta\left(d_n\Delta x_n\right)\right)+p_n\Delta x_{n+1}+q_n f\left(x_{n-\sigma}\right)=0, n\geq n_0$ by means of a Riccati transformation technique.

In [11], the authors considered the oscillatory behavior of solution of third order difference equations of the form

$$\Delta\left(a(n)\left(\Delta^2 x(n)\right)^{\alpha}\right)+q(n)f\left(x[g(n)]\right)=0$$

and $\Delta\left(a(n)\left(\Delta^2 x(n)\right)^{\alpha}\right)=q(n)f\left(x[g(n)]\right)+p(n)h\left(x[\sigma(n)]\right)$ with $\sum^{\infty} a^{-1/\alpha}<\infty$ are established.

In [12], the authors studied the oscillation of neutral type difference equation in third order of the form

$\Delta\left(a_n\left(\Delta^2\left(x_n+p_n x_{n-k}\right)\right)^{\alpha}\right)+q_n f\left(x_{n-l}\right)=0$ by using generalized Riccati type transformation.

The purpose of this paper is to derive some sufficient condition for nonlinear neutral third order oscillatory difference equation (1). The results obtained in this paper have been motivated by that of in [4, 12].

OSCILLATORY RESULTS

Firstly, consider the difference equation

$$\Delta\left(a_e\left(\Delta^2\left(w_e+p_e\, w_{e-l}\right)\right)\right)-q_e f\left(w_{e-m}\right)=0 \qquad (2)$$

is established by giving sufficient conditions for oscillation. We start with some useful lemmas, which we intend to use later. We define,

$$k_e=w_e\pm p_e\, w_{e-l}$$

$$B_e=\sum_{S=e_n}^{\infty} a_s$$

$$T_e=(1-p_{e-m})At_e$$

$$V_e=\sum_{S=e_n}^{e-1} T_s \quad \text{for all} \quad e\in E_o$$

Lemma 1 Let $\{w_e\}$ be the positive solution of (2) then the function

$K_e=w_e+P_e w_{e-l}$. Satisfy the following cases

$case\ (i)\ \ K_e>0, \Delta K_e>0, \Delta^2 K_e>0$

$case\ (ii)\ \ K_e>0, \Delta K_e<0, \Delta^2 K_e>0$

for $e\geq e_1\in N_0$ where e_1 is sufficiently large.

Proof: The proof is omitted because it is found in [3, 4].

Lemma 2 Let $\{w_e\}$ be the positive solution of (2). Let the function $\{k_e\}$ satisfy case (ii) of Lemma 1. If

$$\sum_{e=e_0}^{\infty}\sum_{s=e}^{\infty}\left[\frac{1}{a_e}\sum_{a_e}^{\infty} q_t\right]^{1/\tau}=\infty \qquad (3)$$

then $\lim_{e\to\infty} w_e=\lim_{e\to\infty} k_e=0$.

Proof: The proof is omitted because it is found in [3].

Lemma 3 If $\{w_e\}$ be the positive solution of (2) and the function k_e satisfy Lemma 1 of case (i). Then $\{z_e\}$ be the positive real sequence exist, such that

$$z_e\geq v_e q_e-\sum_{S=e_n}^{\infty} B_s\, z_{s=1} \qquad (4)$$

$$\lim_{e\to\infty}\sup[z_{e+1}B_e]\leq d \qquad (5)$$

for some constant d > 0 and

$$\sum_{e=e_0}^{\infty} T_e<\infty, \sum_{e=e_0}^{\infty} B_e V_{e+1}<\infty \qquad (6)$$

Proof: Let $\{w_e\}$ be the positive solution of (2).

Assume $w_e>0, w_{e-l}>0$ and $w_{e-m}>0$ for any $e\geq e_1\geq e_0$.

Then $k_e>w_e>0$ and satisfy Lemma 1 of case (i) for any $e\geq N\geq e_1$

$$\Delta\left(a_e(\Delta^2 k_e)\right) \leq q_e f(w_{e-m})$$

$$\Delta\left(a_e(\Delta^2 k_e)\right) \leq q_e w_{e-m} A t_e \ , e \geq e_1 \tag{7}$$

$$w_e \geq (1-p_e)k_e \tag{8}$$

From (7) and (8), we have

$$\frac{\Delta\left(a_e(\Delta^2 K_e)\right)}{K_{e-m}} \leq q_e(1-p_{e-m})At_e \tag{9}$$

Define, $$z_e = \frac{\Delta a_e(\Delta^2 k_e)}{k_{e-m}} \tag{10}$$

Then $z_e > 0$ for all $e \geq e_1$ and

$$\Delta z_e = \frac{\Delta a_e(\Delta^2 k_e)}{k_{e-m}} - \frac{\Delta a_{e+1}(\Delta^2 k_{e-1})}{k_{e-m}\, k_{e-m+1}} \Delta k_e$$

Using (9) and (10) in the last inequality, we obtain

$$\Delta z_e \leq q_e(1-P_{e-m})At_e - z_{e+1}\frac{\Delta k_e}{k_{e-m}} \tag{11}$$

$$\Delta z_e \leq q_e T_e - z_{e+1}\frac{\Delta k_e}{k_{e-m}} \tag{12}$$

The monotonicity property of$\{\Delta^2 k_e\}$, we obtain

$$\Delta k_e = \Delta k_{e_0} + \sum_{S=e_0}^{e-1} \Delta^2 k_e \geq \sum_{S=e_0}^{e-1} \Delta^2 k_e$$

$$\Delta k_e \geq \sum_{S=e_0}^{e-1} a_s\left(a_s(\Delta^2 k_s)\right)$$

$$\Delta k_e \geq B_e\left(a_e(\Delta^2 k_e)\right) \tag{13}$$

Using (13) in the inequality (12), we get

$$\Delta z_e - q_e T_e + z_{e+1} B_e \leq 0 \ , e \geq N \tag{14}$$

Taking summation from N to e-1, we obtain

$$z_e \leq z_N + \sum_{s=N}^{e-1} q_s T_s - \sum_{s=N}^{e-1} z_{s+1} B_s \text{ for } e \geq N_0 \tag{15}$$

We claim that, $\sum_{s=N}^{e-1} T_e < \infty$ for all $e \geq N$

$$z_e \leq z_N - \sum_{s=N}^{e-1} q_s T_s$$

and letting limit $e \to \infty$, we obtain $z_e \to -\infty$, which contradicts z_e

Similarly, we can show that

$$\sum_{s=N}^{\infty} B_s z_{s+1} < \infty$$

Now, letting as $e \to \infty$ in (15), we have

$$z_\infty - z_N - \sum_{s=N}^{\infty} q_s T_s + \sum_{s=N}^{\infty} B_s z_{s+1} \leq 0$$

$$z_e \geq v_e q_e - \sum_{s=N}^{\infty} B_s z_{s+1} \ \text{ for } e \geq N$$

Since $T_e > 0$ and $z_e > 0$ for $e \geq N$, we have from (14) that $\Delta z_e < 0$ and

$\lim_{n\to\infty} z_e = R$, for some constant $R > 0$.

$z_e \geq z_{e+1} B_e$

Taking limit supreme, we obtain

$R \geq \lim_{e\to\infty} \sup(z_{e+1}B_e)$

or $$\lim_{e\to\infty} \sup(z_{e+1}B_e) \leq d$$

for some $d > 0$.

This complete the proof.

Theorem 1 Either the solution (2) oscillate or converge to zero as $e \to \infty$, if the following assumption holds

(C_1) $\lim_{e\to\infty} \inf \frac{1}{V_e}\sum_{s=e}^{\infty} B_s V_{s+1} > \frac{\beta^{\beta}}{(\beta+1)^{\beta+\frac{1}{\beta}}}$

(C_2) $\alpha > \frac{\beta^{\beta}}{(\beta+1)^{\beta+\frac{1}{\beta}}}$

(C_3) $\lim_{e\to\infty} \inf \frac{z_e}{V_e} = \mu$, $\quad \mu > 1$

(C_4) $\lim_{e\to\infty} \inf q_e = 1$

Proof: Suppose $\{w_e\}$ is an non-oscillatory solution of (2).

Without loss of generality assume $w_e > 0, w_{e-e} > 0$ and $w_{e-m} > 0$ for any $e \geq e_1 \geq e_0$.

The corresponding $\{k_e\}$ satisfy Lemma 1

Case (i): Let $\{k_e\}$ satisfy Lemma 1 of case (i)

From Lemma 3, $$z_e \geq V_e q_e - \sum_{s=e}^{\infty} B_s z_{s+1}$$

$$\frac{z_e}{V_e} \geq q_e - \frac{1}{V_e}\sum_{s=e}^{\infty} B_s V_{s+1}\frac{z_{s+1}}{V_{s+1}}$$

Taking limit infimum in the above equation,

$$\lim_{e\to\infty} \inf \frac{z_e}{V_e} \geq \lim_{e\to\infty} \inf q_e - \lim_{e\to\infty} \inf \frac{1}{V_e}\sum_{s=e}^{\infty} B_s V_{s+1}\frac{z_{s+1}}{V_{s+1}}$$

From the assumption of the theorem,

$$\mu - \alpha\mu \geq 1 \qquad (16)$$

But $\frac{n}{(n+1)^n} \leq 1$

$$\mu - \alpha\mu \leq \frac{\beta}{(\beta+1)^{\beta}}\frac{1}{\alpha^{\beta}} \qquad (17)$$

From (16) and (17), we get

$$\alpha \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+\frac{1}{\beta}}}$$

Hence our assumption (C_2) contradicts.

Case (ii): If $\{k_e\}$ satisfy Lemma 1 of case (ii), from the condition (3), when $\tau = 1$.

Theorem 2 Assume $\lim_{e\to\infty} Sup\,[B_e(q_{e+1}V_{e+1} + \sum_{s=e+1}^{\infty} B_s V_{s+1})] = \infty \qquad (18)$

then either the solution (2) oscillate or converge to zero as $e \to \infty$.

Proof: Suppose $\{w_e\}$ be the non-oscillatory solution of (2)

Without losing the generality assume $w_e > 0, w_{e-l} > 0$ and $w_{e-m} > 0$ for any $e \geq e_1 \geq e_0$ and $\{w_e\}$ satisfy the cases of Lemma 1.

Case (i): Let $\{k_e\}$ satisfy Lemma 1 of case (i).

From Lemma 3, $z_e \geq V_e q_e - \sum_{s=e}^{\infty} B_s z_{s+1}$

Since $z_e \geq v_e$, we have

$$z_e \geq V_e q_e - \sum_{s=n}^{\infty} B_s V_{s+1}$$

Using this in (4), we get

$$\lim_{e\to\infty} Sup\left[B_e\left(V_{e+1}q_{e+1}+\sum_{s=e+1}^{\infty}B_sV_{s+1}\right)\right]\le d$$

Which is the contradiction.

Case (ii): If $\{k_e\}$ satisfy Lemma 1 of case (ii), from the condition (3), where $\tau = 1$.

We have $\lim_{e\to\infty} w_e = 0$.

The proof is complete.

Secondly, we consider the following difference equation

$$\Delta\left(a_e\left(\Delta^2(w_e - p_e w_{e-l})\right)\right) - q_e f(w_{e-m}) = 0 \qquad (19)$$

The oscillation of its solutions are established by using some sufficient conditions.

Lemma 4 If $\{w_e\}$ be the positive solution of (19) and the function $\{k_e\}$ satisfy Lemma 1 of case (i). Then $\{z_e\}$ be the positive real sequence exist, such that

$$z_e \ge q_e T_e - \sum_{s=e}^{\infty} B_s z_{s+1} \qquad (20)$$

$$\lim_{e\to\infty} Sup\,[z_{e+1}B_e] \le d \qquad (21)$$

For some constant d > 0 and

$$\sum_{e=e_0}^{\infty} T_e < \infty \ , \ \sum_{e=e_0}^{\infty} B_e T_{e+1} < \infty \qquad (22)$$

Proof: Let $\{w_e\}$ be the positive solution of (19).

Assume $w_e > 0, w_{e-l} > 0$ and $w_{e-m} > 0$ for any $e > e_1 > e_0$.

Then $k_e > w_e > 0$ and satisfy Lemma 2.1of case (i), for any $e \ge N \ge e_1$

$$\Delta\left(a_e(\Delta^2 k_e)\right) \le q_e w_{e-m} A t_e$$

There exist two possible cases.

Case (i): $k_e > 0$, similar to the proof of Lemma 3 so we omitted the details.

Case (ii): $k_e < 0$, yet for any $e \ge e_2 \ge e_1 \ge e_0$, then we have two cases for w_e.

Case (a): Assume w_e is unbounded, then

$$w_e = k_e - p_e w_{e-l} < -p_e w_{e-l} < w_{e-l} \qquad (23)$$

Since $\{w_e\}$ is unbounded.

Choose a sequence $\{w_{e_k}\}$ satisfy $\lim_{k\to\infty} w_k = \infty$ where $\lim_{k\to\infty} w_{c_k} = \infty$ and max $w_e = w_{E_e}$

Choosing N to be large such that $l(N_k) > N_1$ for any $N_k > e_2$

Thus $max\ w_e = w_{N_e}$, which is the contradiction to (23).

Case (b): Assume $\{w_e\}$ is bounded.

Show that $w_e \to 0$ as $e \to \infty$, $\qquad \lim_{e\to\infty} Sup\ k_e \le 0$

Then we have, $\qquad \lim_{e\to\infty} \text{Sup}\ (w_e - p_e w_{e-l}) \le 0$

$$\lim_{e\to\infty} \text{Sup}\ w_e - \gamma \lim_{e\to\infty} \text{Sup}\ w_{e-l} \le 0$$

$$(1-\gamma)\lim_{e\to\infty} \text{Sup}\ w_e \le 0$$

This gives $w_e \to 0$ as $e \to \infty$.

The proof is complete.

Theorem 3 Either the solution (19) oscillate or converge to zero as $e \to \infty$, if the following assumption holds

$(C_1)\ \lim_{e\to\infty}\inf\frac{1}{T_e}\sum_{s=e}^{\infty}B_sT_{s+1} > \frac{\beta^\beta}{(\beta+1)^{\beta+\frac{1}{\beta}}}$

$(C_2)\ \alpha > \frac{\beta^\beta}{(\beta+1)^{\beta+\frac{1}{\beta}}}$

$(C_3)\ \lim_{e\to\infty}\inf\frac{z_e}{T_e} = \mu\,, \qquad \mu > 1$

$(C_4)\ \lim_{e\to\infty}\inf q_e = 1$

Proof: Suppose that $\{w_e\}$ of (19) be the non-oscillatory solution.

Without losing the generality assume $w_e > 0, w_{e-e} > 0$ and $w_{e-m} > 0$ for any $e \geq e_1 \geq e_0$.

The corresponding $\{k_e\}$ satisfy Lemma 1

Case (i): Let $\{k_e\}$ satisfy Lemma 1of case (i).

From equation (20) of Lemma 4,

$$\frac{z_e}{T_e} \geq q_e - \frac{1}{T_e}\sum_{s=e}^{\infty}B_sz_{s+1}$$

$$\frac{z_e}{T_e} \geq q_e - \frac{1}{T_e}\sum_{s=e}^{\infty}B_sT_{s+1}\frac{z_{s+1}}{T_{s+1}}$$

Taking limit infimum in the above equation and from the assumption of the theorem,

$$\mu - \alpha\mu \geq 1 \qquad (24)$$

But $\frac{n}{(n+1)^n} \leq 1$

$$\mu - \alpha\mu \leq \frac{\beta}{(\beta+1)^\beta}\frac{1}{\alpha^\beta} \qquad (25)$$

From (24) and (25), we get

$$\alpha \leq \frac{\beta^\beta}{(\beta+1)^{\beta+\frac{1}{\beta}}}$$

Hence our assumption (C_2) contradicts.

Case (ii): If $\{k_e\}$ satisfy Lemma 1 of case (ii), from the condition (3), where $\tau = 1$.

We have $\lim_{e\to\infty} w_e = 0$.

The proof is complete.

Theorem 4 Assume $\lim_{e\to\infty} Sup\,[B_e(q_{e+1}\,T_{e+1} + \sum_{s=e+1}^{\infty}B_s\,T_{s+1})] = \infty$ (26)

then either the solution (19) oscillate or converge to zero as e$\to\infty$.

Proof: Suppose $\{w_e\}$ be the non-oscillatory solution of (19)

Without losing the generality assume $w_e > 0, w_{e-l} > 0$ and $w_{e-m} > 0$ for any $e \geq e_1 \geq e_0$.

The corresponding $\{w_e\}$ satisfy Lemma 1.

Case (i): Let $\{k_e\}$ satisfy Lemma 1 of case (i).

From Lemma 4, $z_e \geq T_eq_e - \sum_{s=e}^{\infty}B_sz_{s+1}$

Since $z_e \geq T_e$, we have

$$z_e \geq T_eq_e - \sum_{s=n}^{\infty}B_sT_{s+1}$$

Using this in (21), we get

$$\lim_{s\to\infty} Sup \left[B_s\left(V_{s+1}q_{s+1} + \sum_{s=s+1}^{\infty} B_s V_{s+1}\right)\right] \leq d$$

Which is the contradiction.

Case (ii): If $\{k_s\}$ satisfy Lemma 1 case (ii), from the condition (3), where $\tau = 1$.

We have $\lim_{s\to\infty} w_s = 0$.

The proof is complete.

CONCLUSION

In this paper, some new oscillation criteria for third order neutral delay difference equation is obtained by utilizing summation average techniques and comparison principal. This study aim is to develop some new criteria of third order oscillatory neutral delay difference equation. So that we apply them when the other criteria fail. In future, we extend this results for higher order neutral delay difference equations.

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REFERENCES

[1] E. Thandapani and K. Mahalingam: Osciilatory properties of third order neutral delay difference equations, Demons. Math., 35(2), 2002, 325-336.
[2] E. Thandapani, M. Vijaya and T. Li; On the oscillation of third order halflinear neutral type difference equations, Elec. J. Qual. Theo. Diff. Eqns., 76, 2011, 1-13.
[3] E. Thandapani and S. Selvarangam: Oscillation results for third order halflinear neutral difference equations, Bull. Math. Anal. Appl., 4, 2012, 91-102.
[4] E. Thandapaniand S. Selvarangam: Oscillation of third order half linear neutral difference equations, Math. Bohemica, 138, 2013, 87–104.
[5] G. Ayyappan: Oscillation Criteria of Third Order Nonlinear Neutral Difference Equations, Malaya J. Mat. 3(2), 2015, 216-223.
[6] M. F. Aktas, A. Tiryaki and Z. Zafer: Oscillation of third order nonlinear delay difference equations, Turk J. Math., 36, 2012, 422-436.
[7] R. P. Agarwal: Difference equations and Inequalities. Theory, Methods and Applications, Second Edition, Marcel Dekker, New York (2000).
[8] R. P. Agarwal, M. Bohner, S. R. Grace and D. O' Regan: Discrete Oscillation Theory. Hindawi Publ. Corp., NewYork (2005).
[9] S. Elayadi: An Introduction to Difference Equations. Third Edition, Springer, New York, (2005).
[10] S. H. Saker, J.O. Alzabut and A.Mukheime: On the oscillatory behavior for a certain third order nonlinear delay difference equations, Elec.J.Qual. Theo. Diff. Eqns., 67, 2010, 1-16.
[11] S. R. Grace, R.P.Agarwal and J. R. Graef: Oscillation criteria for certain third order nonlinear difference equations, Appl. Anal. Discrete Math., 3, 2009, 27-38.
[12] S. Selvarangam, M. Madhan, Ethiraju Thandapani, Sandra Pinelas; Improved Oscillation Conditions for Third-Order Neutral Type Difference Equations, Electronic Journal of Differential Equations, Vol. 2017, No. 90, 2017, 1–13.
[13] Daxue Chen, Oscillation Properties for Second-Order Half-Linear Dynamic Equations on Time Scales, Journal of Mathematics, Research, Vol. 4, No. 2; April 2012.
[14] E. Thandapani, Oscillation theorems for perturbed nonlinear second order difference equations, Computers & Mathematics with Applications, Vol.28, No.1-3, pp.309-316, 1994.
[15] Said R. Grace, Oscillatory Behavior of Third-Order Nonlinear Difference Equations with a Nonlinear-Nonpositive Neutral Term, Mediterranean Journal of Mathematics, 2019