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Solution of Linear Partial Differential Equation with Variable Coefficient by HPM

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Abstract - In the current research work, Homotopy Perturbation method will be used to solve systems of linear partial with variable coefficient differential equations. We will be solving few problems of Linear PDE with Homotopy Perturbation method. The resultant error will be compared in the exact solutions. Based upon the perturbation and decomposition theories, these methods proved to be highly accurate and useful in the current times. To further show the utilization of these methods we will be presenting few examples of variable coefficient with different boundary conditions.

Index Terms - Variable coefficient, PDE, Solution, HPM.

INTRODUCTION

In theory, the Homotopy perturbation method shown by *Ji- Huan* [1,2,3,4] is studied to determine the linear partial differential equation with variable coefficient. The numerical outcomes perceived are definite explanation of the formula.

To understand various physical developments and problems associated with PD equation having variable coefficients in mathematical physics, along with more range of science and engineering, we inspect the correct explanation. Upon further examination we find out that this formula gives us the correct explanation of linear partial differential equation having variable coefficient.

PARTIAL DIFFERENTIAL EQUATION (PDE) WITH VARIABLE COEFFICIENT

The application of PDE is widely used in applied mathematics and science. A differential equation which involves two or more independent variable is called PDE.

It is called linear if the dependent variable and its partial derivatives are only in the first degree and do not multiply together.

A PDE with variable coefficient is introduced when a linear PDE gets multiplied with a variable[7,8]. Like in this equation u is a dependent variable and $x,\ t$ are independent variable.

HOMOTOPY PERTURBATION FORMULA:

Suppose

$$B(u) - g(s)=0, \quad s \in \xi \tag{1.1}$$

along the initial condition of

$$C(u, \frac{\partial u}{\partial n})=0, \quad s \in \lambda, \tag{1.2}$$

where point *B* is a general operative, *g(s)* is an accepted analytic action, C is an initial operative, and λ is the boundary of the domain ξ. The operative *B* may be generally divided further into two operative, *K* and *M*, where point *K* is a linear and *M* a nonlinear operative.

$$K(u) + M(u) - g(s) = 0 \tag{1.3}$$

To prove the homotopy formula, we have created a homotopy v(s, p): $\xi \times [0,1] \rightarrow S$,

$$H(v, p) = (1 - p)[K(v) - K(u_0)] + p[B(v) - g(s)] = 0, \tag{1.4}$$

Or

$$H(v, p) = K(v) - K(u_0) + p\,K(u_0) + p\,K(u_0) + p[M(v) - g(s)] = 0 \tag{1.5}$$

point $p \in [0, 1]$ is called homotopy limitation (5,6,9) and u_0 is an original proximate as the explanation about (1.1), and that produce the initial condition. certainly, about (1.4) or (1.5), we get

$H(v, 1) = K(v) - K(u_0) = 0$ (1.6)

$H(v, 1) = B(v) – g(s) = 0$ (1.7)

Along with altering action like *p* from zero to unity is just that of *H(v, p)* from $K(v) - K(u_0)$ to $B(v) - g(s)$.

In topology, here is called deformative $K(v) - K(u_0)$ and $B(v) - g(s)$ are called homotopy

We proposed that the explanation (1.4) or (1.5) can be written as a series in *p* as pursue:

$$V = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \ldots \quad (1.8)$$

Putting *p =1* result in the proximate explanation as (1.9)

$$U = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots$$

To check the method's efficiency, we have given different examples of linear PDE with variable coefficient.

EXAMPLE 1: Let us consider a linear differential equation with variable coefficient.

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1, \quad t > 0 \quad (1.1)$$

With original situation

$$u(x, 0) = x^4;\ u_t(x, 0) = 0 \quad (1.2)$$

Let us consider the following homotopy.

$$\frac{\partial^2 u}{\partial t^2} = p[\frac{\partial^2 u}{\partial t^2} + \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2} - D^2{}_t\, u] \quad (1.3)$$

Now by using Homotopy Perturbation formula we get.

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + p^4u_4 + \ldots \quad (1.4)$$

Using the equation (1.3) in equation (1.4) and correlating the coefficient various power of *p* we get.

$$p^0 = \frac{\partial^2 u_0}{\partial t^2} = 0; \quad \frac{\partial u_0}{\partial t} = 0; \quad u(x, 0) = u_0 = x^4.$$

$$p^1 = \frac{\partial^2 u_1}{\partial t^2} = \frac{x^4}{2}(u_0)_{xx}$$

$$= \frac{x^2}{2}(12x^2); \quad \frac{\partial u_1}{\partial t} = 6x^4 t; \quad u_1 = x^{2z}\frac{t^2}{2!}$$

$$p^2 = \frac{\partial^2 u_2}{\partial t^2} = \frac{x^2}{2}(u_1)_{xx} \qquad = x^2\frac{t^2}{2!}$$

$$p^2 = u_2 = x^2\frac{t^4}{4!}$$

$$p^3 = u_3 = x^2\frac{t^6}{6!}$$

:

:

Solving above differential equation under the condition $u_t(x, 0) = 0$

$$u_0 = x^2$$

$$u_1 = x^2\frac{t^2}{2!}$$

$$u_2 = x^2\frac{t^4}{4!}$$

$$u_3 = x^2\frac{t^6}{6!}$$

:

:

$u = u_0 + u_1 + u_2 + u_3 + u_{4} + \ldots$

$u = x^2 + x^2\frac{t^2}{2!} + x^2\frac{t^4}{4!} + x^2\frac{t^6}{6!} + \ldots.$

$= x^2 \cos ht$

Which is correct explanation.

Example 2: Let us resolve the Partial Differential equation with variable coefficient.

$$u_{tt} = \frac{x^2}{12} u_{xx} \quad 0 < x < 1,\ t > o \tag{2.1}$$

With fundamental case

$$u(x,0) = 0; \quad u_t(x,0) = x^6 \tag{2.2}$$

present to the HPF.

Let us consider the following homotopy.

$$\frac{\partial^2 u}{\partial t^2} = p[\frac{\partial^2 u}{\partial t^2} + \frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} - D^2{}_t u] \tag{2.3}$$

Now by Homotopy Perturbation formula we get.

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + p^4u_{4} + \ldots \tag{2.4}$$

Using the equation (2.4) in equation (2.3) and comparing the coefficient of various power of p we get,

$p^0 = \frac{\partial^2 u_0}{\partial t^2} = 0; \quad \frac{\partial u_0}{\partial t} = x^6; \quad u(x,0) = u_0 = tx^6.$

$p^1 = \frac{\partial^2 u_1}{\partial t^2} = \frac{x^2}{12}(u_0)_{xx}$

$= \frac{x^2}{12}(30x^4 t); \quad \frac{\partial u_1}{\partial t} = \frac{5}{2}x^6\frac{t^2}{2}; \quad u_1 = \frac{5}{2}x^6\frac{t^3}{3!}$

$p^2 = \frac{\partial^2 u_2}{\partial t^2} = \frac{x^2}{12}(u_1)_{xx}$

$= \frac{25}{4}x^6\frac{t^3}{3!}$

$p^2 = u_2 = \frac{25}{4}x^6\frac{t^5}{5!}$

$p^3 = x^4\frac{t^7}{7!}$

:

:

Solving above differential equation under the condition $u_t(x,0) = x^4$

$u_0 = x^6 t$

$u_1 = \frac{5}{2}x^6\frac{t^3}{3!}$

$u_2 = \frac{25}{4}x^6\frac{t^5}{5!}$

:

:

$u = u_0 + u_1 + u_2 + u_3 + u_{4} + \ldots$

$u = tx^6 + \frac{5}{2}x^6\frac{t^3}{3!} + \frac{25}{4}x^6\frac{t^5}{5!} + \ldots + \ldots = x^6 (5/2)^n \sinh t$

Which is correct explanation.

In this PDE when we applied HPM we get exact solution.

Example 3: Let us proposed partial differential equation with variable coefficient.

$$u_{tt} = \frac{x^2}{6} u_{xx} \quad 0 < x < 1, \quad t > o \tag{3.1}$$

With fundamental situation

$u(x,0) = x^3; \; u_t(x,0) = 0$

Present to the HPF let us consider the following homotopy.

$$\frac{\partial^2 u}{\partial t^2} = p[\frac{\partial^2 u}{\partial t^2} + \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2} - D^2{}_t u] \tag{3.2}$$

Now by homotopy perturbation formula we get.

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + p^4 u_{4+\dots} \tag{3.3}$$

Using the equation (3.3) in equation (3.2) along with correlate the coefficient of various power of p we get.

$p^0 = \frac{\partial^2 u_0}{\partial t^2} = 0; \quad \frac{\partial u_0}{\partial t} = 0; \; u(x,0) = u_0 = x^3.$

$p^1 = \frac{\partial^2 u_1}{\partial t^2} = \frac{x^2}{6}(u_0)_{xx} = \frac{x^2}{6}(6x); \quad \frac{\partial u_1}{\partial t} = x^3 t; \quad u_1 = x^3 \frac{t^2}{2!}$

$p^2 = \frac{\partial^2 u_2}{\partial t^2} = \frac{x^2}{6}(u_1)_{xx} = x^3 \frac{t^2}{2!}$

$p^2 = u_2 = x^3 \frac{t^4}{4!}$

$p^3 = u_3 = x^3 \frac{t^6}{6!}$

:

:

Solving above differential equation under the condition $u_t(x,0) = 0$

$u_0 = x^3$

$u_1 = x^3 \frac{t^2}{2!}$

$u_2 = x^3 \frac{t^4}{4!}$

$u_3 = x^3 \frac{t^6}{6!}$

:

:

$u = u_0 + u_1 + u_2 + u_3 + u_{4+\dots}$

$u = x^3 + x^3 \frac{t^2}{2!} + x^3 \frac{t^4}{4!} + x^3 \frac{t^6}{6!} + \dots = x^3 \cos ht$

Which is correct explanation.

CONCLUSION

In this paper we have solved three problems of Linear PDE using HPM. We can see that obtained solution is same as exact solution. The final results tell us that the proposed method is efficient and easier to handle when compared with the exact solutions or Adomian Decomposition Method (ADM).

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