

Some Applications of r^{th} order Extorial function in Generalized Difference Operator

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Abstract:

In this paper, using the n^{th} order positive and negative extorial function, we derive higher order difference operator values of n^{th} order positive and negative extorial function. Correct examples are inserted to explain the main theorems.

1. Introduction

The difference operator Δ , $\Delta u(k) = u(k+1) - u(k)$ where $\{u(k)\}$ sequence of numbers. The factorial polynomial is $k_l^{(s)} = \prod_{r=0}^{s-1} (k - rl)$. This extorial function verify the difference equation $\Delta_l^r u(k) = v(k)$.

Definition: 1.1

If $l \neq 0$ is any real and $u(k)$ is any real valued function, Then the difference operator on $u(k)$ defined as,

$$\Delta_l u(k) = u(k+l) - u(k)$$

Example : if $u(k) = e^{2k}$,

$$\text{Then } \Delta_l u(k) = e^{2(k+l)} - e^{2k}$$

$$\Delta_l^2 u(k) = e^{2(k+2l)} - 2e^{2(k+l)} + e^{2k}$$

$$\Delta_l^3 u(k) = e^{2(k+3l)} - 3e^{2(k+2l)} + 3e^{2(k+l)} - e^{2k}$$

$$\Delta_l^4 u(k) = e^{2(k+4l)} - 4e^{2(k+3l)} + 6e^{2(k+2l)} - 4e^{2(k+l)} + e^{2k}$$

Definition: 1.2

The generalized polynomial factorial is,

$$k_l^{(s)} = k(k-l)(k-2l) \dots (k-(s-1)l)$$

Lemma: 1.3

For fixed $k \in \mathbb{R}$, $l, r \in \mathbb{N}$, we have

$$(i) \quad \Delta_l (k_l^{(r)}) = rl k_l^{(r-1)}$$

$$(ii) \quad \Delta_l \left(\frac{1}{k_l^{(r)}} \right) = \frac{-rl}{(k+l)_l^{(r+1)}}$$

$$(iii) \quad \Delta_l^{-1} (k_l^{(r)}) = \frac{k_l^{(r+1)}}{l(r+1)} + C$$

2. r^{th} order positive extorial function

Definition: 2.1

If $-1 < l < 1$, k is real, then r^{th} order extorial function is defined as $e_r(k_l) = 1 + \frac{k_l^{(r)}}{r!} + \frac{k_l^{(2r)}}{(2r)!} + \frac{k_l^{(3r)}}{(3r)!} + \dots + \infty$ (1)

Lemma: 2.2

For $k \in \mathbb{R}$, and $l, r \in \mathbb{N}$,

$$i) \quad e_r(-k_l) = \begin{cases} e_r(k_{(-l)}), & \text{if } r \text{ is even} \\ 1 - \frac{k_{(-l)}^{(r)}}{r!} + \frac{k_{(-l)}^{(2r)}}{(2r)!} - \frac{k_{(-l)}^{(3r)}}{(3r)!} + \dots, & \text{if } r \text{ is odd} \end{cases}$$

$$\text{ii) } e_r(-k_{(-l)}) = \begin{cases} e_r(k_{(l)}), & \text{if } r \text{ is even} \\ 1 - \frac{k_l^{(r)}}{r!} + \frac{k_l^{(2r)}}{(2r)!} - \frac{k_l^{(3r)}}{(3r)!} + \dots, & \text{if } r \text{ is odd} \end{cases}$$

Theorem 2.3

Let $k \in \mathbb{N}$ and $r, l \in \mathbb{N}$ and $tr \neq 1, r \neq s$. Then we have $\Delta_l^s(e_r(k_l)) = l^s \sum_{t=1}^{\infty} \frac{k_l^{(tr-s)}}{(tr-s)!}$

Proof:

$$\begin{aligned} \text{We have } e_r(k_l) &= 1 + \frac{k_l^{(r)}}{r!} + \frac{k_l^{(2r)}}{(2r)!} + \frac{k_l^{(3r)}}{(3r)!} + \dots + \infty \\ &= \left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!} \right] \end{aligned}$$

$$\Delta_l(e_r(k_l)) = \Delta_l \left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!} \right]$$

$$\Delta_l(e_r(k_l)) = l \sum_{t=1}^{\infty} \frac{k_l^{(tr-1)}}{(tr-1)!}$$

$$\begin{aligned} \Delta_l^2(e_r(k_l)) &= \Delta_l(\Delta_l e_r(k_l)) \\ &= \Delta_l \left[l \sum_{t=1}^{\infty} \frac{k_l^{(tr-1)}}{(tr-1)!} \right] \end{aligned}$$

$$\Delta_l^2(e_r(k_l)) = l^2 \sum_{t=1}^{\infty} \frac{k_l^{(tr-2)}}{(tr-2)!}$$

Similarly,

$$\Delta_l^3(e_r(k_l)) = l^3 \sum_{t=1}^{\infty} \frac{k_l^{(tr-3)}}{(tr-3)!}$$

$$\Delta_l^4(e_r(k_l)) = l^4 \sum_{t=1}^{\infty} \frac{k_l^{(tr-4)}}{(tr-4)!}$$

In general,

$$\Delta_l^s(e_r(k_l)) = l^s \sum_{t=1}^{\infty} \frac{k_l^{(tr-s)}}{(tr-s)!} \dots\dots\dots(2)$$

Example:2.5

Put $l = 4, r = 2, s = 3$. These values substitute in previous lemma.

$$\begin{aligned} \text{L.H.S in (2)} &= \Delta_4^2(e_3(k_4)) \\ &= \Delta_4^2 \left(1 + \frac{k_4^{(3)}}{3!} + \frac{k_4^{(6)}}{6!} + \frac{k_4^{(9)}}{9!} + \dots + \infty \right) \\ &= 4^2 \left[\frac{k_4^{(1)}}{1!} + \frac{k_4^{(4)}}{4!} + \frac{k_4^{(7)}}{7!} + \dots + \infty \right] \end{aligned}$$

$$\begin{aligned} \text{R.H.S in (2)} &= l^n \sum_{t=1}^{\infty} \frac{k_l^{(tm-n)}}{(tm-n)!} \\ &= 4^2 \sum_{t=1}^{\infty} \frac{k_4^{(3t-2)}}{(3t-2)!} \\ &= 4^2 \left[\frac{k_4^{(1)}}{1!} + \frac{k_4^{(4)}}{4!} + \frac{k_4^{(7)}}{7!} + \dots + \infty \right] \end{aligned}$$

L.H.S = R.H.S

Lemma: 2.6

For any positive s and real $k, r \neq s$. We have $\Delta_l^{-s}(e_r(k_l)) = \frac{1}{l^s} \sum_{t=0}^{\infty} \frac{k_l^{(ts+r)}}{(ts+r)!}$

Proof:

$$\text{We have } e_r(k_l) = \left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!} \right]$$

$$\Delta_l^{-1}(e_r(k_l)) = \Delta_l^{-1} \left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!} \right]$$

$$\Delta_l^{-1}(e_r(k_l)) = \frac{1}{l} \sum_{t=0}^{\infty} \frac{k_l^{(tr+1)}}{(tr+1)!}$$

$$\begin{aligned} \Delta_l^{-2}(e_r(k_l)) &= \Delta_l^{-1} \left(\Delta_l^{-1}(e_r(k_l)) \right) \\ &= \Delta_l^{-1} \left[\frac{1}{l} \sum_{t=0}^{\infty} \frac{k_l^{(tr+1)}}{(tr+1)!} \right] \\ &= \frac{1}{l} \left(\sum_{t=0}^{\infty} \frac{k_l^{(tr+2)}}{l(tr+2)!} \right) \\ &= \frac{1}{l^2} \left(\sum_{t=0}^{\infty} \frac{k_l^{(tr+2)}}{(tr+2)!} \right) \end{aligned}$$

$$\Delta_l^{-2}(e_r(k_l)) = \frac{1}{l^2} \sum_{t=0}^{\infty} \frac{k_l^{(tr+2)}}{(tr+2)!}$$

Similarly,

$$\Delta_l^{-3}(e_r(k_l)) = \frac{1}{l^3} \sum_{t=0}^{\infty} \frac{k_l^{(tr+3)}}{(tr+3)!}$$

$$\Delta_l^{-4}(e_r(k_l)) = \frac{1}{l^4} \sum_{t=0}^{\infty} \frac{k_l^{(tr+4)}}{(tr+4)!}$$

In general,

$$\Delta_l^{-s}(e_r(k_l)) = \frac{1}{l^s} \sum_{t=0}^{\infty} \frac{k_l^{(tr+s)}}{(tr+s)!}, \quad r \neq s$$

3. rth order negative extorial function

Definition: 3.1

If $|l| < 1$ and k is any real, the negative r^{th} order extorial function is

$$e_{(-r)}(k_l) = 1 + \frac{1}{r!} \frac{1}{k_l^{(r)}} + \frac{1}{(2r)!} \frac{1}{k_l^{(2r)}} + \frac{1}{(3r)!} \frac{1}{k_l^{(3r)}} + \dots + \infty$$

Lemma: 3.2

For $k > 0$, $-1 < l < 1$, $s \neq r$ and $\frac{1}{k_l^{(r)}} = k_l^{(-r)}$, we have

$$\Delta_l^s(e_{(-r)}(k_l)) = (-1)^s l^s \sum_{p=1}^{\infty} \frac{(pr+1)(pr+2) \dots (pr+(s-1))}{(pr-1)! (k+l)_l^{(pr+s)}}$$

Proof:

We have $e_{(-r)}(k_l) = 1 + \frac{1}{r!} \frac{1}{k_l^{(r)}} + \frac{1}{(2r)!} \frac{1}{k_l^{(2r)}} + \frac{1}{(3r)!} \frac{1}{k_l^{(3r)}} + \dots + \infty$

$$= \sum_{p=0}^{\infty} \frac{1}{(pr)! k_l^{(pr)}}$$

$$\Delta_l e_{(-r)}(k_l) = \Delta_l \left(\sum_{p=0}^{\infty} \frac{1}{(pr)! k_l^{(pr)}} \right)$$

$$= \left[\sum_{p=1}^{\infty} \frac{-lpn}{pr(pr-1)! (k+l)_l^{(pn+1)}} \right]$$

$$\Delta_l e_{(-r)}(k_l) = -l \sum_{p=1}^{\infty} \frac{1}{(pr-1)! (k+l)_l^{(pn+1)}}$$

$$\Delta_l^2 e_{(-r)}(k_l) = \Delta_l \left[-l \sum_{p=1}^{\infty} \frac{1}{(pr-1)! (k+l)_l^{(pr+1)}} \right]$$

$$\Delta_l^2 e_{(-r)}(k_l) = l^2 \sum_{p=1}^{\infty} \frac{(pr+1)}{(pr-1)! (k+2l)_l^{(pr+2)}}$$

Similarly $\Delta_l^3(e_{(-r)}(k_l)) = -l^3 \sum_{p=1}^{\infty} \frac{(pr+1)(pr+2)}{(pr-1)!(k+3l)_l^{(pr+3)}}$

In general,

$$\Delta_l^s(e_{(-r)}(k_l)) = (-1)^s l^s \sum_{p=1}^{\infty} \frac{(pr+1)(pr+2) \dots (pr+(s-1))}{(pr-1)! (k+sl)_l^{(pr+s)}}$$

Theorem: 3.3

For $-1 < l < 1, r \neq s, k > 0$ and $\frac{1}{k_l^{(r)}} = k_l^{(-r)}$, we have

$$\Delta_l^{-s}(e_{(-r)}(k_l)) = \frac{1}{l^s} \left[\frac{k_l^{(s)}}{s!} + \sum_{p=1}^{\infty} \frac{1}{(pr)! (pr-1)(pr-2) \dots (pr-s)(k-sl)_l^{(pr-s)}} \right]$$

Proof:

We have,

$$e_{(-r)}(k_l) = 1 + \frac{1}{r!} \frac{1}{k_l^{(r)}} + \frac{1}{(2r)!} \frac{1}{k_l^{(2r)}} + \frac{1}{(3r)!} \frac{1}{k_l^{(3r)}} + \dots + \infty$$

$$= \sum_{p=0}^{\infty} \frac{1}{(pr)! k_l^{(pr)}}$$

$$\Delta_l^{-1}(e_{(-r)}(k_l)) = \Delta_l^{-1} \sum_{p=0}^{\infty} \frac{1}{(pr)! k_l^{(pr)}}$$

$$\Delta_l^{-1}(e_{(-r)}(k_l)) = \frac{1}{l} \left[\frac{k_l^{(1)}}{1!} - \sum_{p=1}^{\infty} \frac{1}{(pr)!(pr-1)(k-l)_l^{(pr-1)}} \right]$$

$$\Delta_l^{-2}(e_{(-r)}(k_l)) = \Delta_l^{-1} \frac{1}{l} \left[\frac{k_l^{(1)}}{1!} - \sum_{p=1}^{\infty} \frac{1}{(pr)!(pr-1)(k-l)_l^{(pr-1)}} \right]$$

$$\Delta_l^{-2}(e_{(-r)}(k_l)) = \frac{1}{l^2} \left[\frac{k_l^{(2)}}{2!} + \sum_{p=1}^{\infty} \frac{1}{(pr)!(pr-1)(pr-2)(k-2l)_l^{(pr-2)}} \right]$$

Similarly,

$$\Delta_l^{-3}(e_{(-r)}(k_l)) = \frac{1}{l^3} \left[\frac{k_l^{(3)}}{3!} - \sum_{p=1}^{\infty} \frac{1}{(pr)!(pr-1)(pr-2)(pr-3)(k-3l)_l^{(pr-3)}} \right]$$

In general,

$$\Delta_l^{-s}(e_{(-r)}(k_l)) = \frac{1}{l^s} \left[\frac{k_l^{(s)}}{s!} + (-1)^s \sum_{p=1}^{\infty} \frac{1}{(pr)!(pr-1)(pr-2) \dots (pr-s)(k-sl)_l^{(pr-s)}} \right]$$

Conclusion

The n order positive and negative extorial functions are explained the values of nth order difference operator values. Here we solved the values for the inverse higher order difference operator.

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