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Non-Polynomial Spline Method for Solving Fourth Order Singularly Perturbed Boundary Value Problems

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Abstract: This study targeted at obtaining a numerical solution of fourth order singularly perturbed boundary value problems using a non-polynomial septic spline. Convergence analysis is briefly discussed and the method is exhibited to have sixth order and eighth order convergence. To confirm the relevance of the method, two model illustrations have been figured for different values of the perturbation parameter and mesh sizes. The numerical results have been tabulated and also represented in graphs. Comparisons are made to confirm the reliability and accuracy of the suggested method.

Keywords: Septic Non-Polynomial, Small parameter, Convergence, End condition, Boundary Layer, Monotone Matrix.

1. Introduction

In the past decades, a reasonable number of articles have appeared on non-classical methods that cover highly second-order differential equations [1]. But few authors had been formulated numerical methods for higher-order singularly perturbed differential equations. In recent years, numerical methods such as quadratic non-polynomial splines [1], septic polynomial [2], and quintic polynomial [3] were being developed for solving such types of problems. Further, we interested with fourth-order singularly perturbed boundary value problems as they arise in varios fields of applied mathematics and engineering such as solid mechanics, chemical kinetics, Newtonian fluid mechanics, quantum mechanics, optimal control, chemical reactor theory, aerodynamics, hydrodynamics , geophysics, etc[3, 4]. Due to the effect of perturbation parameter, the solution of singularly perturbed problem is changed rapidly in some parts of the domain and slowly in other parts. The part where the solution fluctuates promptly is called the interior region [5, 6]. In similar way, the solutions of singularly perturbed boundary problem (BVP) acts like a multi-scale character swing swiftly near a thin transition layer, while away from the layer and it performs regularly which varies steadily due to perturbation parameter ε and meshes size h [7, 1]. Consequently, many obstacles encountered in solving singularly perturbed boundary value problems using standard numerical methods and very difficult to treat as compared to non-singularly perturbed problems[8]. Over and above, the numerical treatment of singularly perturbed problems faces major computational difficulties, and most of the classical numerical methods cannot provide accurate results for all independent values of x when ε is very small relative to the mesh size h [1, 9]. Therefore, it is necessary to develop a more accurate numerical method that works agreeably for $\varepsilon << h$, where most of the numerical methods fail to impart the best results. The study intends to develop a new spline method for the solution of the fourth-order singularly perturbed boundary value problem, which is convergent and more accurate than the existing methods which work for the cases where other methods fail to give good results. We consider singularly perturbed reaction-diffusion boundary value problems with the form:

$$Ly(x) \equiv -\varepsilon y^{(4)}(x) + p(x)y(x) = f(x) \qquad (1)$$

$$y(0) = \delta_0,\ y(1) = \delta_1,\ y'(0) = \delta_2,\ y'(1) = \delta_3$$

Where $\delta_0,\ \delta_1,\ \delta_2$ and δ_3 where constant and ε is small positive parameter, also $p(x)$ and $f(x)$ are small functions.

The spline function has the form:

$$T_n = span\{1, x, x^2, x^3, x^4, x^5, sinkx, coskx\}$$

It is to be noted that k can be real or imaginary.

2. Formulation of The Method

We consider a uniform mesh Δ with nodal points xi on the interval $[a, b]$, such that:

$\Delta: a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,\ x_i = x_0 + ih, i = 0, 1 \dots n$ Where$h = \frac{b-a}{h}$.

Non-polynomial spline function $S_\Delta(x)$ of a class $C^6[a, b]$ which interpolates $y(x)$ at the mesh points $x_i, i = 0, 2, \dots, n$ depends on a parameter$k$, if we take $k \to 0$ then it reduces to ordinary septic spline in $[a, b]$. For each segment$[x_i, x_{i+1}]$; $i = 1, 2, \dots, n-1$, we consider the non-polynomial spline form:

$$S_\Delta(x) = l_i cosk(x-x_i) + g_i sink(x-x_i) + f_i(x-x_i)^5 + e_i(x-x_i)^4 + d_i(x-x_i)^3 + c_i(x-x_i)^2 + b_i(x-x_i) + a_i \qquad (2)$$

Where $i = 1, 2, \dots, n-1$, $a_i, b_i c_i d_i, e_i, f_i, g_i$ and l_i are unknown coefficients and k is free parameter which will be used to raise the accuracy of the method. Let $y(x)$ be the exact solution of Eq. (1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $S_\Delta(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}). Then to determine the eight coefficients of integration of Eq. (2) in terms of $y_i, y_{i+1}, T_i, T_{i+1}, F_i, F_{i+1}$, M_i and M_{i+1}.

We define the interpolator conditions at x_i and x_{i+1} as follows:

$$S_\Delta(x_i) = y_i$$

$$S''_\Delta(x_i) = T_i$$

$$S^{(4)}_\Delta(x_i) = F_i$$

$$S^{(6)}_\Delta(x_i) = M_i$$

$$S_\Delta(x_{i+1}) = y_{i+1}$$

$$S''_\Delta(x_{i+1}) = T_{i+1}$$

$$S^{(4)}_\Delta(x_{i+1}) = F_{i+1} \qquad (3)$$

$$S^{(6)}_\Delta(x_{i+1}) = M_{i+1}$$

The coefficients in Eq. (2) using Eq. (3) are determined as follows:

$$a_i = y_i + \frac{M_i}{k^6},$$

$$b_i = \frac{1}{h}(y_{i+1} - y_i) - \frac{h}{6}(T_{i+1} + 2T_i) + \frac{h^3}{360}(7F_{i+1} + 8F_i) + \frac{h^5}{\theta^6}(M_{i+1} - M_i)$$

$$\frac{h^5}{6\theta^4}(M_{i+1} + 2M_i) + \frac{h^5}{360\theta^2}(7M_{i+1} + 8M_i),$$

$$c_i = \frac{T_i K^4 - M_i}{24K^2},$$

$$d_i = \frac{1}{6h}(T_{i+1} - T_i) - \frac{h}{36}(F_{i+1} + 2F_i) - \frac{h^3}{36\varepsilon^2}(M_{i+1} + 2M_i) - \frac{h^3}{6\theta^4}(M_{i+1} - M_i)$$

$$e_i = \frac{F_i K^2 - M_i}{24k^2}, \qquad (4)$$

$$f_i = \frac{1}{120h}(F_{i+1} - F_i) + \frac{1}{120\theta^2}(M_{i+1} - M_i),$$

$$g_i = \frac{M_i \cos(\theta) - M_{i+1}}{k^6 \sin(\theta)},$$

$$l_i = -\frac{M_i}{\theta^6},$$

where $\theta = kh$ and $i = 0, 1, 2, \dots, n.$

Straight-away from the continuity conditions, i.e. the continuity of the first, third, and fifth derivatives at the point (x_i, S_i) where the two septic spline functions $S_{i-1}(x) and\ S_i(x)$ join , we get:

$$\begin{cases} S'_{\Delta-1}(x_i) = S'_\Delta(x_i) \\ S''_{\Delta-1}(x_i) = S''_\Delta(x_i) \\ S'''_{\Delta-1}(x_i) = S'''_\Delta(x_i) \end{cases} \qquad (5)$$

For $i = 1, 2, \dots, n-1$, Eqs. (4) and (5) yields the relations:

$$T_{i-1} + 4T_i + T_{i+1} = \frac{6}{h^2}(y_{i+1} - 2y_i + y_{i-1}) - \frac{h^2}{60}(7F_{i+1} + 16F_i + F_{i-1}) + 6h^4(\alpha_2 M_{i+1} + 2\beta_2 M_i + \alpha_2 M_{i-1} \qquad (6)$$

$$T_{i-1} - 2T_i + T_{i+1} = \frac{h^2}{6}(F_{i-1} + 4F_i + F_{i+1}) + h^4(\alpha_1 M_{i+1} + 2\beta_2 M_i + \alpha_1 M_{i-1} \qquad (7)$$

$$F_{i-1} - 2T_i + F_{i+1} = h^2(\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} \qquad (8)$$

From equations Eqs. (6) – (8), we obtain the following scheme:

$$-\frac{1}{h^4}(y_{i-3} + \beta_1 y_{i-2} + \beta_2 y_{i-1} + \beta_3 y_i + \beta_2 y_{i+1} + \beta_1 y_{i+2} + y_{i+3}) =$$
$$\alpha_1 F_{i-3} + \alpha_2 F_{i-2} + \alpha_3 F_{i-1} + \alpha_4 F_i + \alpha_3 F_{i+1} + \alpha_2 F_{i+2} + \alpha_1 F_{i+3} \qquad (9)$$

Where $i = 3, 4, \dots, n-3$ and

$$\beta_1 = \frac{4\theta + 2\theta\cos(\theta) - 6\sin(\theta)}{\sin(\theta) - \theta},$$
$$\beta_2 = \frac{7\theta + 8\theta\cos(\theta) - 15\sin(\theta)}{\sin(\theta) - \theta},$$
$$\beta_3 = \frac{8\theta + 12\theta\cos(\theta) - 20\sin(\theta)}{\sin(\theta) - \theta},$$
$$\alpha_1 = \frac{120\theta - 20\theta^3 + \theta^5 - 120\sin(\theta)}{120\theta^4(\sin(\theta) - \theta)},$$
$$\alpha_2 = \frac{13\theta^5 - 20\theta^3 - \theta(120 - 20\theta^2 + \theta^4)\cos(\theta) - 360\sin(\theta)}{60\theta^4(\sin(\theta) - \theta)},$$
$$\alpha_3 = \frac{840\theta + 100\theta^3 + 67\theta^5 + (960\theta + 80\theta^3 - 52\theta^5)\cos(\theta) - 1800\sin(\theta)}{120\theta^4(\sin(\theta) - \theta)},$$
$$\alpha_4 = \frac{-7\theta^3 + 600\sin(\theta) - 3\theta(11\theta^4 + 20\theta^2 + 120)\cos(\theta) - 240\theta}{30\theta^4(\sin(\theta) - \theta)}.$$

At nodal point x_i, the proposed singularly perturbed Eq. (1) can be discretized as:

$$-\varepsilon y_i^{(4)} + p y_i = f_i \qquad 10$$

Where $p_i = p(x_i)$ and $f_i = f(x_i)$.

From Eqn. (10), we obtain:

$$y_i^{(4)} = \frac{p_i y_i - f_i}{\varepsilon}$$

By using spline fourth derivatives, we have:

$$\left.\begin{aligned} F_i &= \tfrac{u_i y_i - f_i}{\varepsilon} \\ F_{i-2} &= \tfrac{u_{i-2} y_{i-2} - f_{i-2}}{\varepsilon} \\ F_{i+1} &= \tfrac{u_{i+1} y_{i+1} - f_{i+1}}{\varepsilon} \\ F_{i+3} &= \tfrac{u_{i+3} y_{i+3} - f_{i+3}}{\varepsilon} \end{aligned}\right\}$$

$$\left.\begin{aligned} F_{i-3} &= \tfrac{u_{i-3} y_{i-3} - f_{i-3}}{\varepsilon} \\ F_{i-1} &= \tfrac{u_{i-1} y_{i-1} - f_{i-1}}{\varepsilon} \\ F_{i+2} &= \tfrac{u_{i+2} y_{i+2} - f_{i+}}{\varepsilon} \end{aligned}\right\} \qquad (11)$$

We discretize Eq. (8) at the grid points $x_i, i = 1, 2, \dots, n$ and using Eq. (7), we obtain:

$$(\varepsilon + h^4\alpha_1 p_{i-3})y_{i-3} + (\beta_1\varepsilon + h^4\alpha_2 p_{i-2})y_{i-2} + (\beta_2\varepsilon + h^4\alpha_3 p_{i-1})y_{i-1} + (\beta_3\varepsilon + h^4\alpha_4 p_i)y_i +$$
$$(\beta_2\varepsilon + h^4\alpha_3 p_{i+1})y_{i+1} + (\beta_1\varepsilon + h^4\alpha_2 p_{i+2})y_{i+2} + (\varepsilon + h^4\alpha_1 p_{i+3})y_{i+3} =$$
$$h^4(f_{i-3} + \beta_1 f_{i-2} + \beta_2 f_{i-1} + \beta_3 f_i + \beta_2 f_{i+1} + \beta_3 f_{i-2} + f_{i-3}) + t_i \qquad (12)$$

Where $i = 3, 4, \dots, n-3$ and local truncation error t_i

$$
\begin{aligned}
t_i =\ & \varepsilon(2+2\beta_1+2\beta_2+\beta_3)y_i+\varepsilon(9+4\beta_1+\beta_2)h^2y_i^{(2)}+\frac{1}{12}\varepsilon(16\beta_1+\beta_2+24\alpha_1+24\alpha_2+24\alpha_3+12\alpha_4)h^4y_i^{(4)}\\
&+\frac{1}{360}\varepsilon(64\beta_1+\beta_2+3240\alpha_1+1440\alpha_2+360\alpha_3+729)h^6y_i^{(6)}\\
&+\frac{1}{2016}\varepsilon(256\beta_1+\beta_2+136080\alpha_1+26880\alpha_2+1680\alpha_3+6561)h^8y_i^{(8)}\\
&+\frac{1}{1814400}\varepsilon(1024\beta_1+\beta_2+3674160\alpha_1+322560\alpha_2+5040\alpha_3+59049)h^{10}y_i^{(10)}\\
&+\frac{1}{239500800}\varepsilon(4096\beta_1+\beta_2+77944680\alpha_1+3041280\alpha_2+5040\alpha_3+531441\,)h^{12}y_i^{(12)}
\end{aligned} \tag{13}
$$

Where $i=3,4,\dots,n-3$

3. END CONDITION

Since the Eq. (9) consists $n-5$ equations in $(n-1)$ unknowns, so four more equations are required as end conditions. Consider the end condition in the following form:

$$\sum_{l=0}^{4} b_l y_l + c_1 h y_0' = h^4 \sum_{l=0}^{7} d_1 y_l^{(4)} + t_1, \qquad i=1 \tag{14}$$

$$\sum_{l=0}^{5} v_l y_l + c_2 h y_0' = h^4 \sum_{l=0}^{7} m_1 y_l^{(4)} + t_2, \qquad i=2 \tag{15}$$

$$\sum_{l=0}^{5} v_l y_{n-l} + c_2 h y_n' = h^4 \sum_{l=0}^{7} m_1 y_{n-l}^{(4)} + t_2, \qquad i=n-2 \tag{16}$$

$$\sum_{l=0}^{4} b_l y_{n-1} + c_1 h y_n' = h^4 \sum_{l=0}^{7} d_1 y_{n-l}^{(4)} + t_{n-1}, \qquad i=n-1 \tag{17}$$

Where b_l, c_1, c_2 and d_l are parameters which are to be calculated using the method of undetermined coefficients. Applying Taylor's expansion on Eqs. (14)-(17), we obtain the coefficients and class of different orders as follows:

3.1 Sixth Order Method

For the choice of $\beta_1=114$, $\beta_2=-465$, $\beta_3=700$, $\alpha_1=0$, $\alpha_2=0$, $\alpha_3=-21$, and $\alpha_4=-78$, the local truncating error in the sixth order method is:

$$
t_i=\begin{cases}
-14300\varepsilon h^{10}y_i^{(10)}+O(h^{11}), & i=1, n-1\\
-\frac{123}{12349}\varepsilon h^{10}y_i^{(10)}+O(h^{11}), & i=3,\dots,n-3\\
4120\varepsilon h^{10}y_i^{(10)}+O(h^{11}), & i=2,\ n-2
\end{cases} \tag{18}
$$

Where the coefficients are as follows:

$(b_0,b_1,b_2,b_3,b_4,c_1)=(-8,\frac{161}{3},\frac{191}{3},67,-\frac{103}{6},5)$,

$(v_0,v_1,v_2,v_3,v_4,v_5,c_2)=\left(-8,\frac{161}{3},\frac{191}{3},67,-\frac{103}{6},5\right)=(4,-\frac{19}{3},1,4,-\frac{11}{3},1,2)$

$(d_0,\ d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6,\ d_7)=\left(-\frac{101}{4462},-\frac{807}{209},-\frac{3120}{269},\frac{200}{6549},-\frac{1}{720},0,0\right)$,

$(m_0,m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6,\ m_7)=\left(-\frac{137}{7190},-\frac{925}{3024},\frac{275}{1008},\frac{77}{108},\frac{318}{1823},-\frac{7}{3600},0,0\right)$,

3.2 Eighth Order Method

For the choice of $\beta_1=-\frac{9}{5}$, $\beta_2=-\frac{9}{5}$, $\beta_3=\frac{26}{5}$, $\alpha_1=0$, $\alpha_2=-\frac{193}{120}$, $\alpha_3=-\frac{317}{300}$, and $\alpha_4=-\frac{353}{200}$, the local truncating error in the eighth order method is:

$$t_i = \begin{cases} -3245465021\varepsilon h^{12} y_i^{(12)} + O(h^{13}), & i = 1, n-1 \\ -\frac{7}{74931}\varepsilon h^{12} y_i^{(12)} + O(h^{13}), & i = 3, \dots, n-3 \\ -3247811633\, \varepsilon h^{12} y_i^{(12)} + O(h^{13}), & i = 2,\ n-2 \end{cases} \tag{19}$$

Where the coefficients are as follows:

$(b_0, b_1, b_2, b_3, b_4, c_1) = (-7, \frac{32}{3}, -4, 0, \frac{1}{3}, -4)$,

$(v_0, v_1, v_2, v_3, v_4, v_5, c_2) = \left(-7, \frac{61}{3}, -27, 20, -\frac{22}{3}, 1, -2\right)$

$(d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7) = \left(\frac{191}{5589}, \frac{3554}{4455}, \frac{181}{440}, \frac{762}{5735}, -\frac{163}{2351}, \frac{99}{2798}, -\frac{86}{8477}, \frac{79}{62370}\right)$,

$(m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7) = \left(\frac{89}{4458}, -\frac{183}{952}, \frac{12089}{6157}, \frac{263}{1469}, \frac{267}{2983}, -\frac{164}{3599}, -\frac{175}{12499}, \frac{30}{16753}\right)$,

4. Spline Solution

The spline solution of Eq. (12) with the boundary conditions Eqs. (14) - (17) yields the linear system of order $(n-1) \times (n-1)$ and may be written in matrix form as:

$$\begin{cases} AY = C + T \\ A\bar{Y} = C; \\ A(Y - \bar{Y}) = T \\ AE = T \end{cases} \tag{20}$$

Where $Y = y(x_i)$, $\bar{Y} = \bar{y}(x_i)$, $T = (t_i)$and $E = (e_i), i = 1, 2, \dots, n-1$ are expect approximate, local truncation error and discretization error respectively, also $A$is a matrix of order $(n-1)$ with $A = A0 + h^4 BP$.

The seven band matrix A_0 has the form:

$$A_0 = \begin{pmatrix} b_1\varepsilon & b_2\varepsilon & b_3\varepsilon & b_4\varepsilon & \dots & \dots & \dots & \dots & \dots \\ v_1\varepsilon & v_2\varepsilon & v_3\varepsilon & v_4\varepsilon & v_5\varepsilon & \dots & \dots & \dots & \dots \\ \beta_1\varepsilon & \beta_2\varepsilon & \beta_3\varepsilon & \beta_2\varepsilon & \beta_1\varepsilon & \varepsilon & \dots & \dots & \dots \\ \varepsilon & \beta_1\varepsilon & \beta_2\varepsilon & \beta_3\varepsilon & \beta_2\varepsilon & \beta_1\varepsilon & \varepsilon & \dots & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \dots & \varepsilon & \beta_1\varepsilon & \beta_2\varepsilon & \beta_3\varepsilon & \beta_2\varepsilon & \beta_1\varepsilon & \varepsilon \\ \dots & \dots & \dots & \varepsilon & \beta_1\varepsilon & \beta_2\varepsilon & \beta_3\varepsilon & \beta_2\varepsilon & \beta_1\varepsilon \\ \dots & \dots & \dots & \dots & v_5\varepsilon & v_4\varepsilon & v_3\varepsilon & v_2\varepsilon & v_1\varepsilon \\ \dots & \dots & \dots & \dots & \dots & b_4\varepsilon & b_3\varepsilon & b_2\varepsilon & b_1\varepsilon \end{pmatrix}$$

And the matrix B has the form:

$$\begin{pmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & \dots & \dots \\ m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & \dots & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \dots & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \dots & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\ \dots & \dots & \dots & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 \\ \dots & \dots & m_7 & m_6 & m_5 & m_4 & m_3 & m_2 & m_1 \\ \dots & \dots & d_7 & d_6 & d_5 & d_4 & d_3 & d_2 & d_1 \end{pmatrix}$$

$P = diag(p_i)$, and for the vector $C = (c_1, c_2, \dots, c_{n-2}, c_{n-1})^t$ where:

$$c_i = \begin{cases} -(b_0\varepsilon - h^4 d_0 p_0)\delta_0 - \varepsilon c_0 h\delta_2 - h^4 \sum_{j=0}^{7} b_j f_j, & i = 1, \\ -(v_0\varepsilon - h^4 l_0 p_0)\delta_0 - \varepsilon c_1 h\delta_2 - h^4 \sum_{j=0}^{7} l_j f_j, & i = 2, \\ (\varepsilon + h^4\alpha_1 p_0)\delta_0 + h^4(\alpha_1(f_0 + f_6) + \alpha_2(f_1 + f_5) + \alpha_3(f_4 + f_2) + \alpha_4 f_3 & i = 3, \\ h^4(\alpha_1(f_{i-3} + f_{i+3}) + \alpha_2(f_{i-2} + f_{i+2}) + \alpha_3(f_{I-1} + f_{i+1}) + \alpha_4 f_i & i = 4, \quad \ldots, \quad n-4, \\ -(\varepsilon + h^4\alpha_1 p_n)\delta_1 + h^4(\alpha_1(f_{n-6} + f_n) + \alpha_2(f_{n-1} + f_{n-5}) + \alpha_3(f_{n-4} + f_{n-2}) + \alpha_4 f_{n-3} & n-3, \\ -(v_0\varepsilon - h^4 l_0 p_n)\delta_1 - \varepsilon c_1 h\delta_3 - h^4 \sum_{j=0}^{7} l_j f_{7-j}, & n-2 \\ -(b_0\varepsilon - h^4 d_0 p_n)\delta_1 - \varepsilon c_0 h\delta_3 - h^4 \sum_{j=0}^{7} b_j f_{7-j}, & n-1 \end{cases}$$

5. Convergence Analysis

In this section, we look into the convergence analysis of the sixth-order method of Eq. (12) along with an Eqs. (14)-(17) based on the non-polynomial septic spline. Our primary purpose is to drive bounds on $\| E \|_\infty$. Therefore, we follow lemma's.

If H is a square matrix of order n and $\| E \|_\infty \leq 1$ then $(1 + H)^{-1}$ exists and

$$\| E \|_\infty \leq \frac{1}{1 - \| E \|_\infty}.$$

Proof: Detail proof is given on [10]

Lemma 2:

The matrix A, given by Eq. (20) is monotone, if $\| p \|_\infty < \frac{2304}{\bar{u}}$, where $\bar{u} = 25(b-a)^4 + 10(b-a)^2 h^2 + 9h^4$

Proof:

From Eq. (20) we have:

$$A = A_0 + h^4 BP$$

It was shown in [11, 12] that $A0$ is non-singular and its inverse satisfies the in equality:

Type equation here.

$$\| A^{-1} \|_\infty \leq \frac{5(b-a)^4 + 10(b-a)^2 h^2 + 9h^4}{384h^4} = O(h^{-4}) \quad (21)$$

Therefore $A = A_0 + h^4 BP = (I - A_0^{-1} h^4 BP)A_0 =$

So to prove singularity of A, it is sufficient to show $I - A_0^{-1} h^4 BP$ is monotone.

Moreover, $\| P \|_\infty \leq \| p \|_\infty = max_{a \leq x \leq b} |p(x)|$[12, 13].

By Cauchy Schwartz and triangle inequalities we may have the following relations [14]:

$$\|\mathrm{A}_0^{-1} \mathrm{h}^4 \mathrm{BP} \|_\infty \leq \|A_0^{-1} \|_\infty \|h^4 BP \|_\infty,$$

$$\leq \|A_0^{-1} \|_\infty h^4 \|BP \|_\infty,$$

$$\leq \|A_0^{-1}\|_\infty h^4 \|B\|_\infty \|P\|_\infty,$$

$$\leq \|A_0^{-1}\|_\infty h^4 \|B\|_\infty \|p\|, \tag{22}$$

Where $\|P\|_\infty = m_0 + m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 = \frac{5}{6}$, therefore, substituting $\|A_0\|_\infty, \|B\|_\infty$ and , $\|P\|_\infty$ in Eq. (22), we get the following relation:

$$\|\mathrm{A}_0^{-1}\mathrm{h}^4\mathrm{BP}\|_\infty \leq 1 \tag{23}$$

From Lemma 1 and Eq. (23), it shows that the matrix A is monotone.

Since A is monotone matrix, Eq. (20) can be written as:

$$E = A^{-1}T = (A_0 + h^4BP)^{-1}A_0T.$$

It follows that,

$$E = A^{-1}T = (I + A^{-1}BP)^{-1}A_0T,$$

and using Cauchy Schwartz inequality we obtain [14]:

$$\|E\|_\infty \leq \|(I + A_0^{-1}h^4BP)^{-1}\|_\infty \|A_0^{-1}\|_\infty, \|T\|_\infty \tag{24}$$

So by using Eq. (24) follow that:

$$\|T\|_\infty \leq \frac{\|A_0^{-1}\|_\infty \|T\|_\infty}{1 - \|\mathrm{A}_0^{-1}\mathrm{h}^4\mathrm{BP}\|_\infty}$$

Using Lemma 1and Eq. (24) we obtain:

$$\|T\|_\infty \leq \frac{\|A_0^{-1}\|_\infty \|T\|_\infty}{1 - \mathrm{h}^4 \|A_0^{-1}\|_\infty \|\mathrm{B}\|_\infty \|p\|} \tag{25}$$

From Eq. (18) we have:

$\|T\|_\infty = -\frac{123}{12349}\varepsilon h^4 M_{10}$, where $M_{10} = max_{a\leq x_i\leq b}\left|y_i^{(10)}(x_i)\right|$ then $|E|_\infty \leq M_{10}h^6$ where M_{10} is constant independent of h.

From Eq. (25) it follows $|E|_\infty = O(h^6)$.

Also from Eq. (19) we have:

$|T|_\infty = \frac{7}{74931}\varepsilon h^{12} M_{12}$, where $M_{12} = max_{a\leq x_i\leq b}\left|y_i^{(12)}(x_i)\right|$ then $|E|_\infty \leq M_{12}h^8$ where M_{12} is constant independent of h.

From Eq. (25) it follows $|E|_\infty = O(h^8)$.

6. Numerical Examples

To demonstrate the validity of the methods, two singularly perturbed problems have been considered. These examples have been chosen because they have been widely discussed in the literature and their exact solutions were available for comparison.

Example 1

Consider the following singularity perturbed problem:

$$-\varepsilon y^{(4)}(x) + p(x) + p(x)y(x) = f(x), \quad 0 \le x \le 1$$
$$y(0) = 1 \quad y(1) = 0 \quad y'(0) = 0, \qquad y'(1) = 0$$

Where:

$$f(x) = (x-1)^4 x^8 \sin(\varepsilon x) - \varepsilon x^4(-16\varepsilon^3(x-1)^3 x^3(3x-2)\cos(\varepsilon x) + 96\varepsilon x(14 - 84x + 18x^2 - 165x^3 + 55x^4)(14 - 44x + 33x^2)\sin(\varepsilon x) + 24(70 - 504x + 1260x^2 - 1320x^3 + 495x^4)\sin(\varepsilon x))$$

and the analytic solution is

$y(x) = (1-x)^4 x^8 \sin(\varepsilon x)$.

Example 2

Consider the following singularly perturbed problem:

$$-\varepsilon y^{(4)}(x) + p(x) + p(x)y(x) = f(x), \quad 0 \le x \le 1$$
$$y(0) = 0 \quad y(1) = 0 \quad y'(0) = 0, \qquad y'(1) = 0$$

$$f(x) = \varepsilon x^4 \Big(32\varepsilon^2 x\big(-6(7 - 55x^2 + 70x^8) + \varepsilon^2(x^2 - 2x^{10})\big)\cos(\varepsilon x) + (x^4(x^4-1)^2 - \varepsilon^5 x^4(x^4-1)^2) + 48\varepsilon^3 x^2(7 - 33x^4 + 30x^8) - 240\varepsilon(7 - 99x^4 + 182x^8)\Big)\sin(\varepsilon x))$$

and the analytic solution is:

$$y(x) = \varepsilon x^8 (x^4 - 1)^2 \sin(\varepsilon x).$$

The numerical solutions in terms of maximum absolute errors and comparison with the literatures are given in tables 1 and 2 with its graphically in Figures 1 and 2 with different value of the perturbation parameters ε and mesh size$N$. On the top of this the rates of convergence are calculated for both examples using eighth order by the formula:

$$r^N = log_2 \frac{E^N}{E^{2N}},$$

With r is rate of convergence.

Table 1: Maximum absolute errors for example 1.

$\varepsilon \downarrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
New Method (order Eight)				
$\frac{1}{16}$	$8.3243e-10$	$4.0618e-12$	$1.6310e-14$	$2.0439e-16$
$r \rightarrow$	3.9206	4.2018	4.0025	
$\frac{1}{32}$	$4.3058e-10$	$2.1018e-12$	$8.4112e-15$	$6.3696e-17$
$r \rightarrow$	3.9200	3.7701	3.6865	
$\frac{1}{64}$	$2.3082e-10$	$1.1253e-12$	$4.5328e-15$	$4.3370e-17$
$r \rightarrow$	3.9218	3.8822	3.6148	
$\frac{1}{128}$	$1.3489e-10$	$6.5701-13$	$2.6339e-15$	$1.6566e-17$
$r \rightarrow$	4.1090	4.2041	4.0727	

New Method (Order Six)				
$\frac{1}{16}$	$4.3469e-08$	$8.1929e-10$	$1.3215e-11$	$2.0775e-13$
$\frac{1}{32}$	$2.2394e-08$	$4.2244e-10$	$6.8162e-12$	$1.0716e-13$
$\frac{1}{64}$	$1.1910e-08$	$2.2548e-10$	$3.6324e-12$	$5.6858e-14$
$\frac{1}{128}$	$6.9096e-09$	$1.3057e-10$	$2.1029e-12$	$3.2944e-14$
Reference [1]				
$\frac{1}{16}$	$1.0499e-07$	$9.8529e-09$	$7.0265e-10$	$4.6045e-11$
$\frac{1}{32}$	$5.3745e-08$	$5.0610e-09$	$3.6108e-10$	$2.3663e-11$
$\frac{1}{64}$	$2.8376e-08$	$2.6766e-09$	$1.9132e-10$	$1.2538e-11$
$\frac{1}{128}$	$1.6215e-08$	$1.5345e-09$	$1.0968e-10$	$7.1910e-12$
Reference [2]				
$\frac{1}{16}$	$1.6660e-06$	$1.3100e-07$	$2.6140e-09$	$6.7160e-11$
$\frac{1}{32}$	$8.5370e-07$	$6.7360e-08$	$1.3440e-09$	$3.4520e-11$
$\frac{1}{64}$	$4.5200e-07$	$3.5690e-08$	$7.1280e-10$	$1.8290e-11$
$\frac{1}{128}$	$2.6000e-07$	$2.0490e-08$	$4.0920e-10$	$1.0500e-11$

Table 2: Maximum absolute errors for example 2.

$\varepsilon \downarrow$	$N=16$	$N=32$	$N=64$	$N=128$
New Method (order Eight)				
$\frac{1}{16}$	$5.8608e-09$	$4.5359e-11$	$1.9667e-13$	$8.4636e-16$
$r \rightarrow$	3.2551	4.0910	4.0031	
$\frac{1}{32}$	$1.5146e-09$	$1.1735e-11$	$5.0792e-14$	$2.9923e-16$
$r \rightarrow$	3.2535	3.6570	3.66875	
$\frac{1}{64}$	$4.0480e-10$	$3.1425e-12$	$1.3620e-14$	$1.2284e-17$
$r \rightarrow$	3.9218	4.0916	4.4770	
$\frac{1}{128}$	$1.1766e-10$	$9.1600e-13$	$3.9642e-15$	$2.1532e-18$
$r \rightarrow$	4.3486	4.3229	5.2086	
New Method (Order Six)				
$\frac{1}{16}$	$4.4469e-07$	$8.4796e-09$	$1.3765e-10$	$2.1594e-12$
$\frac{1}{32}$	$1.1496e-07$	$2.1897e-09$	$3.5635e-11$	$5.5729e-13$
$\frac{1}{64}$	$3.0776e-08$	$5.8481e-10$	$9.5656e-12$	$1.4864e-13$
$\frac{1}{128}$	$8.9689e-09$	$1.6973e-10$	$2.8095e-12$	$4.3224-14$

Reference [3]				
$\frac{1}{16}$	$1.7094e-04$	$4.7425e-05$	$1.2094e-05$	$3.0303e-06$
$\frac{1}{32}$	$4.4022e-05$	$1.2203e-05$	$3.1120e-06$	$7.7974e-07$
$\frac{1}{64}$	$1.1706e-05$	$3.2459e-06$	$8.2662e-07$	$2.0714e-07$
$\frac{1}{128}$	$-$	$-$	$-$	$-$

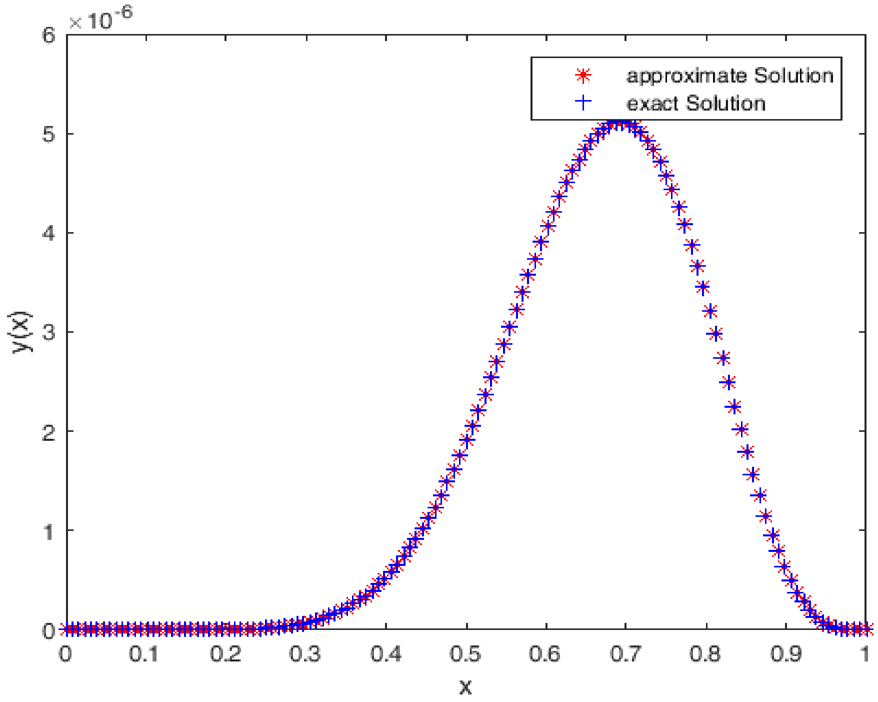

Figure 1: The graph of exact and numerical solution of example 1 for $N = 128$ and $\varepsilon = \frac{1}{64}$.

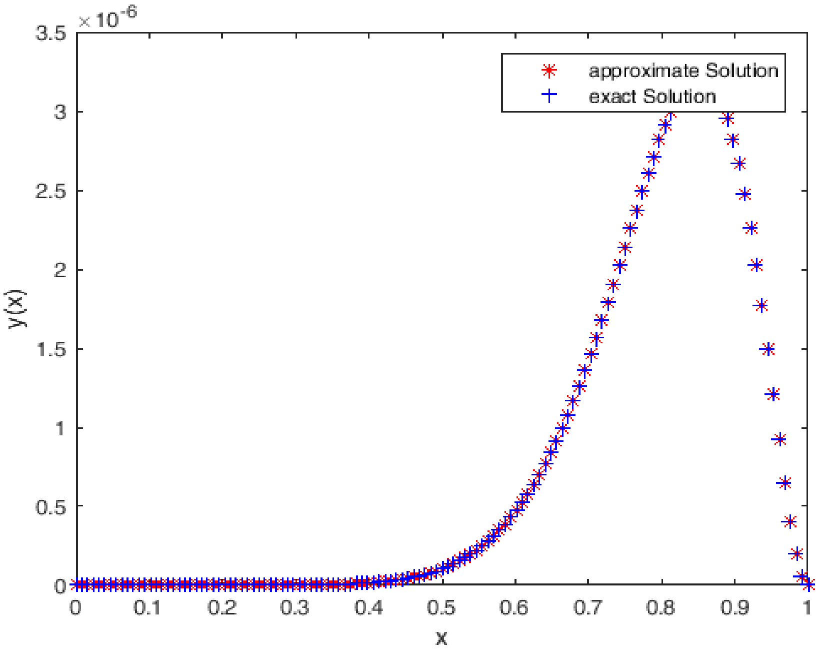

Figure 2: The graph of exact and numerical solution of example 2 for $N = 128$ and $\varepsilon = \frac{1}{128}$.

7. Conclusion

A Non-Polynomial spline method is presented for fourth-order singularly perturbed boundary value problems. Tables 1 and 2 reflect the superiority of the method when the numerical solutions are compared with other approaches. Stability, Convergence, and accuracy are the other power full side of the technique. But also free from complexity to grasp easily when compared with other methods makes the new one more stiff. In the same manner, the agreement on the numerical solution with the analytic solution is mirrored in Figures 1 and 2 in the condition of $\varepsilon \ll h$. The converging test is investigated and demonstrates that the present technique was in sixth and eighth order converged.

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