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Bounded variation solutions of a functional integral equation in $L_1(R^+)$

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Abstract: This paper treats the existence theorem of a functional integral equation in the space of locally bounded variation on an unbounded interval. The concept of measure of noncompactness and a fixed point theorem due to Darbo are the main tools in carrying out our proof.

Keywords: Nemytskii operator, Measure of noncompactness, Functions of bounded variation, Darbo fixed point theorem.

1. Introduction

Integral equations play an important role in the nonlinear analysis and their applications in the theory of elasticity, engineering, mathematical physics and contact problems (see [1], [13], [14], [18]). For instance, the most frequently investigated integral equations are Fredholm linear equation or its nonlinear counterparts, Hammerstein and urysohn integral equation (see [2], [4], [9], [14], [19], [20]).

In this paper we study existence of at least one solution of the functional integral equation

$$x(\tau) = f\big(\tau, x(\tau)\big), \qquad \tau \in R^+ \tag{1}$$

in the space of bounded variation.

2. Preliminaries

In the following we will deal some notations and results that will be needed in the sequel. Let R be the field of real numbers and R^+ be the interval $[0,\infty)$. Denote by $L_1 = L_1(R^+)$ the space of Lebesgue integrable functions in the interval $[0,\infty)$, with the standard norm

$$\|x\| = \int_0^\infty |x(\tau)|\, d\tau.$$

A most important operator in nonlinear analysis is the so-called Nemytskii operator [3].

Definition 2.1 If $f(\tau,x) = f: I \times R \to R$ satisfies Carathéodory conditions i. e. it is measurable in t for any $x \in R$ and continuous in x for almost all $\tau \in R^+$. Then to every function $x(\tau)$ being measurable on R^+ we may assign the function

$$(F_f x)(\tau) = f\big(\tau, x(\tau)\big), \qquad \tau \in I$$

The operator F_f is called the Nemytskii (or superposition) operator generated by *f*.

Also, we present a theorem that gives the necessary and sufficient condition so that the Nemytskii operator maps continuously the space L_1 into itself.

Theorem 2.1 [3] If f satisfies Carathéodory conditions, then the Nemytskii operator F generated by the function f maps continuously the space L_1 into itself if and only if

$$|f(\tau,x)| \le a(\tau) + b|x|,$$

for every $\tau \in R^+$ and $x \in R$, where $a(\tau) \in L_1$ and $b \ge 0$ is a constant.

In the following, we present some definitions and results which will be needed further on. Assume that $(E, \|.\|)$ is an arbitrary Banach space with zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $b(\theta, r)$. If X is a subset of E, then $\bar{X}$ and $ConvX$ denote the closure and convex closure of X, respectively. We denote the standard algebraic operations on sets by the symbols λX and $X + Y$. Moreover, we denote by M_E the family of all nonempty and bounded subsets of E and N_E its subfamily consisting of all relatively compact subsets.

Now, we present the concept of a regular measure of noncompactness:

Definition 2.2 [6]

The mapping $\mu: M_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(i) $\mu(X) = 0 \Leftrightarrow X \in N_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

(iii) $\mu(\bar{X}) = \mu(convX) = \mu(X)$.

(iv) $\mu(\lambda X) = |\lambda| \mu(X)$ for $\lambda \in R$.

(v) $\mu[X + Y] \le \mu(X) + \mu(Y)$.

(vi) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1,2,\cdots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \cup_{n=1}^{\infty} X_n$ is nonempty.

Definition 2.3 [6]

The Hausdorff measure of noncompactness $\chi(X)$ (see also [15], [16]) is defined as

$$\chi(X) = \inf\{r > 0: \text{there exists a finite subset } Y \text{ of } E \text{ such that } x \subset B_r\}.$$

A more general regular can be defined as the space [5]:

$$c(X) = \lim_{\varepsilon\to 0}\{\sup_{x\in X}\{\sup[\int_D |x(\tau)| d\tau: D \subset R^+, \text{ meas}D \le \varepsilon]\}\} = 0 \qquad (2)$$

and

$$d(X) = \lim_{T\to\infty}\{\sup[\int_T^\infty |x(\tau)|\, d\tau: x \in X]\}, \qquad (3)$$

where meas$D$ denotes the Lebesgue measure of a subset D.

Put

$$\gamma(X) = c(X) + d(X). \qquad (4)$$

Then we have the following theorem [17], which connects between the two measures $\chi(X)$ and $\gamma(X)$.

Theorem 2.2 Let $X \in M_E$ and compact in measure, then

$$\chi(X) \le \gamma(X) \le 2\chi(X).$$

Now, we give Darbo fixed point theorem (cf. [8]).

Theorem 2.3 If Q is nonempty, bounded, closed and convex subset of E and let $A: Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists a constant $k \in [0,1)$ such that

$$\mu(AX) \le k\mu(X),$$

for any nonempty subset X of Q. Then A has at least one fixed point in the set Q.

Definition 2.4 (Functions of bounded variation) [7]

Let $x: [a, b] \to R$ be a function. For each partition $P: a = \tau_0 < \tau_1 < \cdots < \tau_n = b$ of the interval $[a, b]$, we define

$$Var(x,[a,b]) = \sup \sum_{i=1}^{n} |x(\tau_i) - x((\tau_{i-1})|,$$

where the supremum is taken over all partitions P of the interval $[a,b]$. If $Var(x) < \infty$, we say that x has bounded variation and we write $x \in BV$.

We denote by $BV = BV[a,b]$ the space of all functions of bounded variation on $[a,b]$.

Theorem 2.4 [5] Assume that $x \subset L_1(I)$ is of locally generalized bounded variation, then $Conv\ X$ $(convex\ hull\ of\ X)$ and $\bar{X}$ are of the same type.

Corollary 2.1 [5] Let $x \subset L_1(I)$ is of locally generalized bounded variation then $Conv\ X$ is also such.

Next, we will have the following theorem that we will be used further on (cf. [5]).

Theorem 2.5 Assume that $x \subset L_1$ is a bounded set have have the following hypotheses:

(i) There exists $\tau_0 \geq 0$ such that the set $x(\tau_0): x \in X$ is bounded on R,
(ii) X is of locally generalized bounded variation on R^+,

(iii) for any $a > 0$ the following equality holds

$$\lim_{T\to\infty} \{\sup_{x\in X}\{meas\{\tau > T: |x(\tau)| \geq a\}\}\} = 0.$$

Then the set X is compact in measure.

Corollary 2.2 [5] If $x \subset L_1(I)$ is a bounded set satisfy the hypotheses of Theorem 2.5. Then $ConvX$ is compact in measure.

3. Main result

Equation (1) takes the form

$$x = Fx, \tag{5}$$

where F is the Nemytskii operator.

We shall treat equation (1) with the following hypotheses listed below:

(i) $f: R^+ \times R \to R$ satisfies Carathéodory conditions and $\exists$ a constant $b \geq 0$ and a function $a \in L_1(R^+)$ such that

$$|f(\tau,x)| \leq a(\tau) + b|x|, \quad \text{for all } \tau \in R^+ \text{ and } x \in R.$$

(ii) $\exists k > 0$ such that

$$|f(\tau,x) - f(\tau,y)| \leq k|x-y|.$$

Moreover, there exists a constant $M > 0$ such that $\forall n \in N$, every partition $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$, the following inequality holds:

$$\sum_{i=1}^{n} |f(\tau_i, x_{i-1}) - f(\tau_{i-1}, x_{i-1})| \leq M.$$

(iii) $b < 1$.

Theorem 3.1 If the hypotheses (i)–(iii) are satisfied, then equation (1) has at least one solution $x \in L_1(R^+)$ which is a function of locally bounded variation on R^+.

Proof. From hypothesis (i) and Theorem 2.1 the operator F maps $L_1(R^+)$ into $L_1(R^+)$ and is continuous.

Also, we get

$$\begin{aligned}\|Fx\| &= \int_0^\infty |f(\tau, x(\tau))|\, d\tau \\ &\leq \int_0^\infty |a(\tau)| d\tau + b \int_0^\infty |x(\tau)| d\tau \\ &\leq \|a\| + b\|x\| \\ &\leq \|a\| + b.r \leq r.\end{aligned}$$

From the previous inequality, the operator F transforms the ball B_r into B_r, where

$$r = \frac{\|a\|}{1-b} > 0.$$

Next, let us choose an $x \in B_r$. In view of assumption (i), we have

$$\begin{aligned} |(Fx)(0)| &= \left|f\big(0, x(0)\big)\right| \\ &\le a(0) + b|x(0)| \\ &< \infty. \end{aligned} \tag{6}$$

Then we get all functions belonging to FB_r are bounded.

Moreover, fix $T > 0$ and consider the sequence τ_i such that $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$. Therefore, we get

$$\sum_{i=1}^{n} |(Fx)(\tau_i) - (Fx)((\tau_{i-1})| = \sum_{i=1}^{n} |f(\tau_i, x(\tau_i)) - f(\tau_{i-1}, x(\tau_{i-1}))|$$

$$\begin{aligned} &\le \textstyle\sum_{i=1}^{n} |f(\tau_i, x(\tau_i)) - f(\tau_i, x(\tau_{i-1}))| \\ &+ \textstyle\sum_{i=1}^{n} |f(\tau_i, x(\tau_{i-1})) - f(\tau_{i-1}, x(\tau_{i-1}))| \\ &\le k \textstyle\sum_{i=1}^{n} |x(\tau_i) - x(\tau_{i-1})| + M \end{aligned}$$

$$V(Fx, T) \le kV(x, T) + M < \infty \tag{7}$$

In view of the above estimate all functions belonging to FB_r have variation majorized by the same constant on every closed subinterval of the interval R^+.

Now, let the set $Q_r = \operatorname{conv} GB_r$, obviously $Q_r \subset B_r$ and the operator F maps Q_r into itself. In view of Theorem 2.1 we deduce that the operator F is continuous on the set Q_r. Moreover, in view of (6), (7) and Theorem 2.5 we deduce that the set FB_r is compact in measure. Also, the set Q_r is compact in measure by using Corollary 2.2. In addition to, the set $Q_r$is of locally generalized bounded variation on $R^+$by using Corollary 2.1.

Now, we prove that the operator F is a contraction with respect to the measure of noncompactness χ.

Let us take a subset X of Q_r and $\varepsilon > 0$ is fixed, then $\forall x \in X$ and for a set $D \subset R^+, \operatorname{meas} D \le \varepsilon$, we have

$$\begin{aligned} |(Fx)(\tau)| &= \left|f\big(\tau, x(\tau)\big)\right| \\ &\le a(\tau) + b|x(\tau)|, \end{aligned}$$

then

$$\int_D |(Fx)(\tau)| d\tau \le \int_D |a(\tau)| d\tau + b \int_D |x(\tau)| d\tau.$$

Also, using the fact that

$$\lim_{\varepsilon \to 0} \sup\{\textstyle\int_D a(\tau) d\tau : D \subset R^+, \operatorname{meas} D \le \varepsilon\} = 0$$

By using definition (2), we get

$$c(FX) \le bc(X). \tag{8}$$

Moreover, fixing $T > 0$ we get

$$\int_T^\infty |(Fx)(\tau)| d\tau \le \int_T^\infty |a(\tau)| d\tau + b \int_T^\infty |x(\tau)| d\tau.$$

As $T \to \infty$, the previous inequality implies

$$d(FX) \le bd(X), \tag{9}$$

where $d(X)$ has been defined before in (3).

Hence from (8) and (9) we obtain

$$\gamma(FX) \le b\gamma(X).$$

where γ denotes the measure of noncompactness defined in (4).

Since X is a subset of Q_r and Q_r is compact in measure, then we obtain

$$\chi(FX) \le b\chi(X).$$

Hence by using hypothesis (iii) allows us to apply Darbo fixed point theorem. This completes the proof. ■

Next, we will treat equation (1) for $\tau \in (0,1)$ as follows:

Theorem 3.2 The equation $x(\tau) = f\big(\tau, x(\tau)\big),\ \tau \in (0,1)$ has at least one solution $x \in L_1(0,1)$ that is a function of locally bounded variation if the following hypotheses

(i) $f: (0,1) \times R \to R$ satisfies Carathéodory conditions and $\exists$ a constant $b \geq 0$ and a function $a \in L_1(0,1)$ such that

$$|f(\tau,x)| \leq a(\tau) + b|x|, \quad \text{for all } \tau \in (0,1) \text{ and } x \in R.$$

(ii) $\exists$ a constant $k > 0$ such that

$$|f(\tau,x) - f(\tau,y)| \leq k|x-y|.$$

Moreover, there exists a constant $M > 0$ such that for every $n \in N$, every partition $\varepsilon = \tau_0 < \tau_1 < \cdots < \tau_n = 1 - \varepsilon$ of (0,1), the following inequality holds:

$$\sum_{i=1}^{n} |f(\tau_i, x_{i-1}) - f(\tau_{i-1}, x_{i-1})| \leq M.$$

(iii) $b < 1$,

are satisfied.

Proof. The proof takes similar steps as Theorem 3.1 so, it is omitted.

4. Uniqueness of the solution

Now, we can prove the existence of our unique solution.

Theorem 4.3 If the hypotheses of Theorem 3.1 is satisfied but instead of assumption (iii), let $k < 1$. Then equation (1) has a unique solution on R^+.

Proof. To prove the unique solution of equation (1), let $x(\tau), y(\tau)$ be any two solutions of equation (1) in B_r, we have

$$\begin{aligned}
\|x-y\| &= \big\|f\big(\tau, x(\tau)\big) - f(\tau, y(\tau))\big\| \\
&= \int_0^\infty \big|f\big(\tau, x(\tau)\big) - f(\tau, y(\tau))\big|\, d\tau \\
&\leq \int_0^\infty |x(\tau) - y(\tau)| d\tau \\
&\leq k\|x-y\|.
\end{aligned}$$

Therefore,

$$(1-k)\|x-y\|_{L_1} \leq 0,$$

This yields that $\|x-y\| = 0, \Rightarrow x = y$, this completes the proof.

5. Example

Assume that the integral equation

$$x(\tau) = e^{-\tau} + \frac{\tau x(\tau)}{\tau+2}, \qquad \tau \in R^+ \tag{10}$$

We have $f(\tau,x) = e^{-\tau} + \frac{\tau x(\tau)}{\tau+2}$, so we can see that f satisfies Carathéodort conditions i.e. it is measurable in τ and continuous in x, where the exponential is continuous and so that it is measurable and the polynomial function is continuous. Also, we get

$$\begin{aligned}
|f(\tau,x)| &= e^{-\tau} + \frac{\tau x(\tau)}{\tau+2} \\
&\leq e^{-\tau} + \frac{1}{3}|x(\tau)|.
\end{aligned}$$

Hence, $a(\tau) = e^{-\tau} \in L_1(R^+)$ and $b = \frac{1}{3} > 0$, then condition (i) is satisfied.

Also,

$$|f(\tau, x) - f(\tau, y)| \leq \frac{1}{2}|x - y|,$$

so that condition (ii) is satisfied. Finally, we have $b = \frac{1}{3} < 1$ then condition (iii) is satisfied. So, our hypotheses of Theorem 3.1 be satisfied, hence equation (10) has at least one solution $x \in BV$ on R^+.

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