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New builds on pseudo topological vector spaces ($T^{\not\approx}{}_{\tilde{p}VS}$ Spaces)

Intesar Harbi[1] and AL-Nafie Z. D. [2*]

[1,2]University of Babylon, College of education for pure science, Iraq.

Abstract. This work deals with introducing a new notion of manifold, that is pseudo manifold building on pseudo topological vector spaces (which are separated, locally convex, pseudo metrizable, p-paracompact, equable, p-regular and admissible spaces).

INTRODUCTION Pseudo topological vector spaces introduced in 1966 by Frölicher, A. & Walter B [1] also several definitions have been given of pseudo topological vector spaces [2], [3]. Here we have adopted the definition presented in [4]. We also adopted the definition of the derivability for the functions defined between these spaces, which were presented in the research [5]. This study deals with how to construct the manifold on the pseudo topological vector space and discusses some issues related to this construction, as well as opened a wide field for studying partition of unit, which has wide applications in mathematics.

Keyword: Filter, $\tilde{C}^{\infty}$chart, $\tilde{C}^{\infty}$atlas and Pseudo topological manifold.

PRELIMINARIES

Definition 3.1[1]: Let $\varXi$ be vector space over R, a filter in $\varXi$ is a nonempty set $\mathfrak{F}$ of subsets of $\varXi$, such that: The empty set is not belonging to $\mathfrak{F}$; if $X_1 \in \mathfrak{F}$ and $X_1 \subset X_2 \Rightarrow X_2 \in \mathfrak{F}$;and if $X_1, X_2 \in \mathfrak{F} \Rightarrow X_1 \cap X_2 \in \mathfrak{F}$. Let $\mathrm{F}(\varXi)$ denote the system of all filters on $\varXi$.

Definition 3.2[1]: A filter-basis in $\varXi$ is a nonempty set β of subsets of $\varXi$ that satisfies the conditions: $\emptyset \notin$ β and, for all $\mathcal{B}_1, \mathcal{B}_2 \in$ β there exists $\mathcal{B}_3 \in$ β such that $\mathcal{B}_1 \cap \mathcal{B}_2 \supset \mathcal{B}_3$.

Definition 3.3[1]:The filter $\mathfrak{F}$ is said to be generated by a filter-basis β if it consists of all subsets of $\varXi$ containing a set from β, and denoted as: $\mathfrak{F}$ = [β] = {A ⊆ $\varXi$: $B \in$ A, $B \in$ β}, and for any $a \in \varXi, [a]$ is a filter denoted by: $[a] = \{A \subseteq \varXi: a \in A\}$.

Definition 3.4[1]: A pseudo topology τ on $\varXi$ is a map $\tau: \varXi \to 2^{\mathrm{F}(\varXi)}$, $m \mapsto \tau(m)$If $\mathfrak{F} \in \tau(m)$, we say that $\mathfrak{F}$ converges to m in τ and we write $\mathfrak{F} \downarrow_m$ (Or $\mathfrak{F} \in \tau(m)$)and the next conditions hold for any filters $\mathfrak{F}$ and p:

(1) $[m] \downarrow_m$, where $[m] = \{\mu \subset \varXi, m \in \mu\}$;
(2) $\mathfrak{F} \downarrow_m , \mathfrak{F} \subseteq$ p $\Rightarrow$ p $\downarrow_m$;
(3) $\mathfrak{F} \downarrow_m$, p $\downarrow_m \Rightarrow \mathfrak{F} \cap$ p $\downarrow_m$.

We simply write $\mathfrak{F} \downarrow$ instead of $\mathfrak{F} \downarrow_0$.

Definition 3.5[1]: The pseudo topological vector space (briefly, T_{PVS}) can be defined in a natural way. A T_{PVS} is a vector space together with a compatible pseudo topology on it. The pseudo topology is called compatible with the vector space structure if the two maps ($\varXi \times \varXi \xrightarrow{+} \varXi$, $R \times \varXi \xrightarrow{\cdot} \varXi$) are continuous.

Definition 3.6[3]: Let $(\varXi, \tau)$ be a T_{PVS} and a set $\mathcal{A} \subseteq \varXi$ can be call open set if $\mathcal{A} = int(\mathcal{A})$. The interior $int(\mathcal{A})$ of a set $\mathcal{A} \subseteq \varXi$ can be denoted as: $int(\mathcal{A}) = \{m \in \mathcal{A}: \mathfrak{F} \in \tau(m)$ implies $\mathcal{A} \in \mathfrak{F}\}$.

Definition 3.7[3]: Let $(\varXi, \tau)$ be a T_{PVS} and $\mathcal{A} \subseteq \varXi$, then $\mathcal{A}$ is closed set if

$\mathcal{A} = \bar{\mathcal{A}}$, $\bar{\mathcal{A}} =$ (CL $\mathcal{A}$) = $\{m \in \varXi : \exists$ p $\downarrow_m \varXi$ and $\mathcal{A} \in$ p$\}$.

Definition 3.8[2]: Let $\varXi$ a vector space over R. Then a map $\|.\|_p: \varXi \to R^+, x \to \|x\|_p$ is said to be a pseudo-seminorm (in brief, PSN) iff the following properties are satisfied: For each $p \in \mathbb{N}$, and for all $x, y \in \varXi$,

For all $\alpha \in R - \{0\}, |\alpha| \le 1 \Longrightarrow \|\alpha.x\|_p \le \|x\|_p$; $\|x + y\|_p \le \|x\|_p + \|y\|_p$;

And $\|x\|_p > 0$.

Definition 3.9[2]: A pseudo metric space is a set $\varXi$ together with a function

$\eth: \varXi \times \varXi \to R^+$ which is satisfies: $\forall\, x, y, z \in \varXi$

$\eth(x, y) > 0$; $\eth(x, y) = \eth(y, x)$;and $\eth(x, y) + \eth(y, z) \ge \eth(x, z)$.

Definition 3.10[2]:A pseudo topological vector space $\varXi$ is said to be pseudo-metrizable if its topology induced by the pseudo metric $\eth: \varXi \times \varXi \to R^+ : \forall x, y \in \varXi, \eth(x, y) \coloneqq \sum_{p=1}^{\infty} 2^{-p} \frac{\|x-y\|_p}{1+\|x-y\|_p}$. Such that $\{\|x\|_p\}_{p\in\mathbb{N}}$ is a countable family of pseudo seminorms on Ξ. It is clear that $\eth$ is a translation invariant. The collection of all open balls: $B^{p}_{1/r}(x) = \left\{y \in \Xi : \|x-y\|_p < \frac{1}{r}\right\}, x \in \Xi, r \in \mathbb{N}/\{0\}$ and $p \in \mathbb{N}$, is a base for the pseudo topology of Ξ.

Definition 3.11[1]: Let $\mathfrak{F}$ be a filter in Ξ. The filter $\mathfrak{F}$ is said to be equable if $\mathfrak{r}\mathfrak{F} = \mathfrak{F}$, such that $\mathfrak{r}$ is a filter neighborhood of a zero.

Definition 3.12[1]: A T_{PVS} Ξ is called equable if and only if for each filter $\mathfrak{F}$ with

$\mathfrak{F} \downarrow$ ($\exists$ an equable filter p) p $\geq \mathfrak{F}$ with p $\downarrow$ Ξ, $(\Xi^{\#}, \tau^{\#})$ is equable pseudo topological vector space.

Definition 3.13[1]: A T_{PVS} (Ξ, τ) is called a separated, If $\mathfrak{F} \downarrow_m \Xi$ and $\mathfrak{F} \downarrow_n \Xi$, then $m = n$ [i.e. if $m \neq n$, then $\tau(m) \cap \tau(n) = \emptyset$] .

Definition 3.14[1]: A $T_{\tilde{p}VS}$ Ξ is called admissible if and only if it satisfies the condition: If $\mathfrak{F} \downarrow \Xi$ then, $CL(\mathfrak{F}) \downarrow \Xi$.

Notation 3.15[5]: A separated locally convex and pseudo metrizable $T_{\tilde{p}VS}$ is $T^{\#}{}_{\tilde{p}VS}$ if it is equable and admissible space. And from now on we suppose that $\Xi, \Xi_1, \Xi_2, \ldots$ are always $T^{\#}{}_{\tilde{p}VS}$.

Definition 3.16[1]:Let (Ξ_1, τ_1), (Ξ_2, τ_2) be a $T^{\#}{}_{\tilde{p}VS}$. The map $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is 1-differentiable at the point a if there exist a map $\ell \in \underline{L}(\Xi_1; \Xi_2)$ such that the map r defined by $f(a+h) = f(a) + \ell(h) + r(h)$ is remainder. The map $\ell \in \underline{L}(\Xi_1; \Xi_2)$ is uniquely determined, it is then called the derivative of f at the point a, and it is denoted by: $\ell = \acute{f}(a)$.

Definition 3.17[5]:Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be a $T^{\#}{}_{\tilde{p}VS}$, a map $f: \Xi_1 \to \Xi_2$ is m-differentiable at a point a, then: $D^m f(a) = D(D^{m-1} f(a))$for all $m \in \mathbb{N}$.

Definition 3.18[5]: The map f between Ξ_1 and Ξ_2 is $\tilde{C}^{\infty}(\Xi_1, \Xi_2)$, smooth map if it is $\tilde{C}^{m}(\Xi_1, \Xi_2)$ for all $m \in \mathbb{N}$. Define $\tilde{C}^{\infty}(\Xi_1, \Xi_2)$ to be the projective limit of this system, $\tilde{C}^{\infty}(E_1, E_2) = \bigcap_{m=0}^{\infty} \tilde{C}^{m}(\Xi_1, \Xi_2)$ [$\mathcal{F} \downarrow \tilde{C}^{\infty}(\Xi_1, \Xi_2)$ iff $\mathcal{F} \downarrow \tilde{C}^{m}(\Xi_1, \Xi_2)$ for all $m \in \mathbb{N}$. $\tilde{C}^{\infty}(\Xi_1, \Xi_2) \subset \tilde{C}^{m}(\Xi_1, \Xi_2) \subset \tilde{C}^{0}(\Xi_1, \Xi_2)$].

Definition 3.19[5]: Let Ξ_1, Ξ_2 are $T^{\#}{}_{\tilde{p}VS}$, and let $\mathbb{f}: \Xi_1 \to \Xi_2$ be a bijection map, we say that $\mathbb{f}$ is a diffeomorphism if $\mathbb{f}$ and $\mathbb{f}^{-1}$are differentiable of class$\tilde{C}^{\infty}(\Xi_1, \Xi_2)$.

Definition 3.20[5]: Let M be a pseudo topological space. A chart (or coordinate chart) (u, α) in M is called $\tilde{C}^{\infty}$chart if α is a diffeomorphism map of u onto $\alpha(u)$ open subset of some $T^{\#}{}_{\tilde{p}VS}$ Ξ.

4-Main result:

Definition 4.1: Let (u, α) and (v, β) are two $\tilde{C}^{\infty}$charts on M such that $u \cap v \neq \emptyset$ then the composite map $\beta \circ \alpha^{-1}: \alpha(v \cap v) \to \beta(u \cap v)$ (called transition map from α to β) is a composition of diffeomorphism, and is itself a diffeomorphism.

Definition 4.2: Two $\tilde{C}^{\infty}$charts (u, φ), (v, θ) of a pseudo manifold are smooth compatible if the two maps:

$\theta \circ \varphi^{-1}: \varphi(u \cap v) \to \theta(u \cap v)$, $\varphi \circ \theta^{-1}: \theta(u \cap v) \to \varphi(u \cap v)$ are smooth of class $\tilde{C}^{\infty}$.

Definition 4.3: Let $\mathcal{A} = \{(u_i, \alpha_i)\} i \in I$ be a collection of $\tilde{C}^{\infty}$ charts on M. We call $\mathcal{A}$ an atlas of class $\tilde{C}^{\infty}$(or $\tilde{C}^{\infty}$atlas) if the following conditions are satisfied:

1- $\bigcup_{i \in I} u_i = M$.
2- The sets of the form $\alpha_i(u_i \cap u_j)$ for $i, j \in I$ are all open in Ξ.
3- Whenever $u_i \cap u_j$ is not empty, the map

$\alpha_j \circ \alpha_i{}^{-1}: \alpha_i(u_i \cap u_j) \to \alpha_j(u_i \cap u_j)$ is a $\tilde{C}^{\infty}$diffeomorphism.

Definition 4.4: Two $\tilde{C}^{\infty}$atlases A and A_1 on M are compatible iff every $\tilde{C}^{\infty}$chart of one is compatible with the other $\tilde{C}^{\infty}$atlas.

This is equivalent to saying that the union $A \cup A_1$is still a smooth atlas.

And a $\tilde{C}^{\infty}$chart (u, α) is compatible with smooth atlas $A = \{(u_a, \alpha_a)_{a \in I}\}$ if and only if $A \cup (u, \alpha)$ is a $\tilde{C}^{\infty}$atlas.

Definition 4.5: A smooth atlas $\mathcal{A}$ on M is maximal if every $\tilde{C}^{\infty}$chart that is smoothly compatible with every $\tilde{C}^{\infty}$chart in $\mathcal{A}$ is already in $\mathcal{A}$.

Definition 4.6: A pseudo manifold of class $\tilde{C}^{\infty}$ is a pair (M,$\mathcal{A}$) where M is a pseudo topological manifold and $\mathcal{A}$ is a smooth structure (maximal $\tilde{C}^{\infty}$atlas) on M.

Remark 4.7:If $\mathcal{A}$ is some $\tilde{C}^{\infty}$atlas for pseudo manifold of class $\tilde{C}^{\infty}$ M, and (U, ϕ) is a smooth chart in $\mathcal{A}$, for any , nonempty open subset, $V \subseteq U$, we get a chart, $(V, \phi/_V)$, and it is clear that this $\tilde{C}^{\infty}$chart is compatible with $\mathcal{A}$. Thus, $(V, \phi/_V)$ is also a chart for pseudo manifold of class $\tilde{C}^{\infty}M$. This observation clarifies that if U is any open subset of a smooth M, then U is also a smooth whose $\tilde{C}^{\infty}$charts are the restrictions of $\tilde{C}^{\infty}$charts on M to U.

Proposition 4.8: Let M be a pseudo manifold of class $\tilde{C}^{\infty}$, and let (u_1, α_1) be a smooth compatible with (u_2, α_2) and (u_2, α_2) is smooth compatible with (u_3, α_3) in a $\tilde{C}^{\infty}$ atlas $\mathcal{A}$, then (u_1, α_1) be a smooth compatible with (u_3, α_3) on $\alpha_1(u_1 \cap u_2 \cap u_3)$.

Proof: Since (u_1, α_1) be a smooth compatible with (u_2, α_2) then the map

$\alpha_2 \circ {\alpha_1}^{-1}: \alpha_1(u_1 \cap u_2) \to \alpha_2(u_1 \cap u_2)$ is smooth and, since (u_2, α_2) be a smooth compatible with (u_3, α_3) then the map

$\alpha_3 \circ {\alpha_2}^{-1}: \alpha_2(u_2 \cap u_3) \to \alpha_3(u_2 \cap u_3)$ is smooth, and the open set $u_1 \cap u_2 \cap u_3$ is subset of $u_1 \cap u_2$ and subset of $u_2 \cap u_3$, and the map $\alpha_3 \circ {\alpha_1}^{-1} = (\alpha_3 \circ {\alpha_2}^{-1}) \circ (\alpha_2 \circ {\alpha_1}^{-1})$ is composite smooth maps then its smooth map on $\alpha_1(u_1 \cap u_2 \cap u_3)$, hence (u_1, α_1) be a smooth compatible with (u_3, α_3) on $\alpha_1(u_1 \cap u_2 \cap u_3)$.

Lemma 4.9: Let $\mathcal{A} = \{(u_\delta, \xi_\delta)\}$ be an $\tilde{C}^\infty$atlas on a pseudo manifold of class $\tilde{C}^\infty M$. If two $\tilde{C}^\infty$charts (υ, ψ) and (ω, φ) are both smooth compatible with the $\tilde{C}^\infty$atlas, $\mathcal{A} = \{(u_\delta, \xi_\delta)\}$, then they are smooth compatible with each other.

Proof: Let $p \in \upsilon \cap \omega$. We need to show that $\varphi \circ \psi^{-1}$ is smooth at $\psi(p)$. Since $\mathcal{A} = \{(u_\delta, \xi_\delta)\}$ is an $\tilde{C}^\infty$ −atlas for M, $p \in u_\delta$ for some δ. Then, $p \in \upsilon \cap \omega \cap u_\delta$. By the proposition above, $\varphi \circ \psi^{-1} = (\varphi \circ {\xi_\delta}^{-1}) \circ (\xi_\delta \circ \psi^{-1})$ is smooth on $\psi(\upsilon \cap \omega \cap u_\delta)$, hence at $\psi(p)$. Since p is an arbitrary point of $\upsilon \cap \omega$, this proves that $\varphi \circ \psi^{-1}$is smooth on $\psi(\upsilon \cap \omega)$. Similarly, $\psi \circ \varphi^{-1}$is smooth on $\varphi(\upsilon \cap \omega)$. Therefore, (υ, ψ) and (ω, φ) are both smooth compatible.

Proposition 4.10: Let $u \subset M$ be open set. If $\alpha: u \to \alpha(u) \subset \Xi$ is a diffeomorphism onto its image, then (u, α) is a smooth chart in the smooth atlas of pseudo manifold of class $\tilde{C}^\infty$ M.

Proof: Let (u_a, α_a) be a smooth chart in smooth atlas of M, then $\alpha \circ {\alpha_a}^{-1}$ and $\alpha_a \circ \alpha^{-1}$ are smooth maps. Then, (u, α) is smooth compatible with the smooth atlas. And from maximality of the smooth atlas of M, then the smooth chart (u, α) is in the smooth atlas of pseudo manifold of class $\tilde{C}^\infty M$.

Definition 4.11: Let $f: M \to R$ is smooth function (or $\tilde{C}^\infty$ differentiable) on M if and only if the composite function $f \circ \alpha^{-1}: v \to R$ is smooth such that

$\alpha: u \to v \subset \Xi$, Ξ is $T^{\#}{}_{\tilde{p}VS}$.

Definition 4.12: Let M and N be pseudo manifold of class $\tilde{C}^\infty$on a $T^{\#}{}_{\tilde{p}VS}$ Ξ_1 and Ξ_2 respectively and let $F: M \to N$ be a map. If every point $p \in M$ has a neighborhood u such that the restriction Flu is smooth, then F is smooth. Conversely, if F is smooth, then its restriction to any open subset of M is smooth.

And we can say F is smooth if for every point $p \in M$ there exists a smooth chart (u, θ) at p and a smooth chart (v, μ) at $F(p)$ such that $F(u) \subset v$ and

$\mu \circ F \circ \theta^{-1}$ is smooth map from M into N.

Example 4.13: Any open subset V of a pseudo manifold of class $\tilde{C}^\infty$ M is also a pseudo manifold. If $\{(u_\delta, \theta_\delta)\}$ is a smooth atlas for M, then $\{(u_\delta \cap V, \theta_\delta|_{u_\delta \cap V})\}_{\delta \in I}$ is a smooth atlas for V , where $\theta_\delta|_{u_\delta \cap V}: u_\delta \cap V \to \Xi$ denotes the restriction of θ_δ to the subset $u_\delta \cap V$.

Lemma 4.14: Let M be a pseudo topological manifold.

1- Every $\tilde{C}^\infty$atlas for M is contained in a unique maximal $\tilde{C}^\infty$atlas.
2- Two $\tilde{C}^\infty$atlases for a pseudo manifold of class $\tilde{C}^\infty$ M determine the same maximal $\tilde{C}^\infty$atlas iff their union is a $\tilde{C}^\infty$atlas.

Proof:1- Let $\mathcal{A}$ be a $\tilde{C}^\infty$atlas for M, and let A denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that A is a

smooth atlas, we need to show that any two charts of A are compatible with

each other, which is to say that for any $(u, \theta), (v, \varphi) \in A$,

$\varphi \circ \theta^{-1}: \theta(u \cap v) \to \varphi(u \cap v)$ is smooth. Let $x = \theta(p) \in \theta(u \cap v)$ be arbitrary.

Because the domains of the charts in $\mathcal{A}$ cover M, there is some chart $(w, \pi) \in \mathcal{A}$ such that $p \in w$. Since every chart in A is smoothly compatible with (w, π), both the maps $\pi \circ \theta^{-1}$ and $\varphi \circ \pi^{-1}$ are smooth where they are defined. Since, $p \in u \cap v \cap w$, it follows that $\varphi \circ \theta^{-1} = (\varphi \circ \pi^{-1}) \circ (\pi \circ \theta^{-1})$ is smooth on a neighborhood of x. Thus $\varphi \circ \theta^{-1}$ is smooth in a neighborhood of each point in $(u \cap v)$. Therefore A is a smooth atlas. Now to check that it is maximal, just note that any chart that is smoothly compatible with every chart in A must be smoothly compatible with every chart in $\mathcal{A}$, so it is already in A. This proves the existence of a maximal smooth atlas containing $\mathcal{A}$.

If B is any other maximal smooth atlas containing A, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $B \subset A$. By maximality of B, $B = A$.

2- Let, $\mathcal{A}_1, \mathcal{A}_2$ two $\tilde{C}^\infty$atlases for a pseudo manifold M,

and let $\mathcal{A}_1 \cup \mathcal{A}_2$ be $\tilde{C}^\infty$atlas, since $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}_1 \cup \mathcal{A}_2$ then $\mathcal{A}_1 \cup \mathcal{A}_2$ is maximal of $\mathcal{A}_1$ and $\mathcal{A}_2$.

The other direct: Suppose $\mathcal{A}_1 \cup \mathcal{A}_2$ is $\tilde{C}^\infty$ maximal atlas then every atlas on a pseudo manifold M is contain in $\mathcal{A}_1 \cup \mathcal{A}_2$, from 1 the maximal $\tilde{C}^\infty$ atlas is unique then $\mathcal{A}_1$, $\mathcal{A}_2$ are specify the same maximal $\tilde{C}^\infty$atlas.

Lemma 4.15: Let $\{(u_i, \theta_i)\}$ is a $\tilde{C}^\infty$atlas for M. If $f: M \to \Xi$ is a map such that $f \circ {\theta_i}^{-1}$is smooth for each i, then f is also smooth.

Proof: We need to prove that $f \circ {\theta_i}^{-1}$ is smooth for any $\tilde{C}^\infty$chart (u, θ) on M. It suffices to show it is smooth in a neighborhood of each point $x = \theta(x_0) \in \theta(u)$. For any $x_0 \in u$, there is a chart (u_i, θ_i) in the atlas whose domain contains x_0. Since $(u, \theta)$is smoothly compatible with (u_i, θ_i), the transition map $\theta_i \circ \theta^{-1}$ is smooth on its domain of definition, which includes x. Thus $f \circ \theta^{-1} = \left(f \circ {\theta_i}^{-1}\right) \circ (\theta_i \circ \theta^{-1})$ is smooth in a neighborhood of x.

Lemma 4.16: Let $M_1, M_2$and M_3 are any three-pseudo manifold of class $\tilde{C}^\infty$. And if the two maps $\mathbb{F}: M_1 \to M_2$ and $\mathbb{G}: M_2 \to M_3$ are smooth maps then, $\mathbb{G} \circ \mathbb{F}$ is also smooth map.

Proof: Let (u, θ) and (v, ϑ) be any $\tilde{C}^\infty$ −charts for M_1 and M_3 respectively. We need to show that $\mathbb{G} \circ \mathbb{F}$ is smooth map (i.e. $\vartheta \circ (\mathbb{G} \circ \mathbb{F}) \circ \theta^{-1}$ is smooth on its domain of definition, namely on $\theta(u \cap (\mathbb{G} \circ \mathbb{F})^{-1}(v))$.For any point $x \in u \cap (\mathbb{G} \circ \mathbb{F})^{-1}(v)$, there is a chart (z, μ) for M_2 such that $\mathbb{F}(x) \in z$. By smoothness of $\mathbb{F}$ and $\mathbb{G}$ we get $\mu \circ \mathbb{F} \circ \theta^{-1}$ and $\vartheta \circ \mathbb{G} \circ \mu^{-1}$ are smooth on its domain of definition, and therefore $\vartheta \circ (\mathbb{G} \circ F) \circ \theta^{-1} = (\vartheta \circ \mathbb{G} \circ \mu^{-1}) \circ (\mu \circ \mathbb{F} \circ \theta^{-1})$ is smooth.

Lemma 4.17: Let M_1, M_2 be any pseudo manifold of class $\tilde{C}^\infty$, then the product space $M_1 \times M_2$ has smooth structure such that the projection maps $\pi_1: M_1 \times M_2 \to M_1, \pi_2: M_1 \times M_2 \to M_2$ are smooth. $M_1 \times M_2$ is called the product pseudo manifold of class $\tilde{C}^\infty$.

Proof: Let $\{(u_i, \theta_i)_{i\in I}\}$, $\{(v_j, \vartheta_j)_{j\in J}\}$be smooth atlases for a two pseudo manifolds M_1, M_2 respectively. Then $\{(u_i \times v_j)_{(i,j)\in I\times J}\}$ is an open cover of $M_1 \times M_2$ and the coordinate smooth map $\theta_i \times \vartheta_j: u_i \times v_j \to \theta_i(u_i) \times \vartheta_j(v_j)$ define a smooth atlas. The compatibility and smoothness of structure are verified, hence $M_1 \times M_2$ is pseudo manifold of class $\tilde{C}^\infty$.

Definition 4.18: $\mathbb{F}: M_1 \to M_2$ is called a local diffeomorphism if every point $x \in M_1$ has a neighborhood u such that $\mathbb{F}(u)$ is open in M_2 and $\mathbb{F}: u \to \mathbb{F}(u)$ is a diffeomorphism.

Definition 4.19:Let M_1 and M_2 are connected pseudo manifold of class $\tilde{C}^\infty$, a smooth covering map of class $\tilde{C}^\infty$, $\rho: M_1 \to M_2$ is a smooth bijective map with the property that every $p \in M_2$ has a neighborhood u such that each component of $\rho^{-1}(u)$ is mapped diffeomorphically on to u by ρ, we will also say that u is evenly covered. The manifold M_2 is called the base of the covering, and M_1 is called a covering space of M_2.

Definition 4.20: Let $\rho: M_1 \to M_2$ is any continuous map, a section of ρ is a continuous map $\sigma: M_2 \to M_1$ such that, $\rho \circ \sigma = Id_{M_2}$.

Definition 4.21: Let $\rho: M_1 \to M_2$ is any continuous map, a local section is a continuous map $\sigma: u \to M_1$ defined on some open set $u \subset M_2$ and satisfying the relation $\rho \circ \sigma = Id_u$.

Lemma 4.22: Let $\rho: M_1 \to M_2$ is a smooth covering map. Every point of $M_1$is in the image of a smooth local section of ρ.For any $y \in M_1$, there is a neighborhood u of $x = \rho(y)$ and a smooth local section $\sigma: u \to M_1$ such that $\sigma(x) = y$.

Proof: Let $u \subset M_2$ be an evenly covered neighborhood of x. If v is the component of $\rho^{-1}(u)$ containing y, then $\rho/_v: v \to u$ is a diffeomorphism (by hypothesis). It follows that $\sigma = (\rho/_v)^{-1}: u \to v$is a smooth local section of ρ and such that, $\sigma(x) = y$.

Proposition 4.23: Suppose $\rho: M_1 \to M_2$ is a covering map of class $\tilde{C}^\infty$ and M_3 is any pseudo manifold of class $\tilde{C}^\infty$. A map $F: M_2 \to M_3$ is smooth if and only if $F \circ \rho: M_1 \to M_3$ is smooth:

Proof: The first direction is clear. Suppose conversely that $F \circ \rho$ is smooth and let $x \in M_2$ be arbitrary. By the preceding lemma, there is a neighborhood u of x and a smooth local section $\sigma: u \to M_1$, so that $\rho \circ \sigma = Id_u$.Then the restriction of F to u satisfies: $F/_u = F \circ Id_u = F \circ (\rho \circ \sigma) = (F \circ \rho) \circ \sigma$; which is a composition of smooth maps. Thus F is smooth on u. Since F is smooth in a neighborhood of each point, it is smooth.

Lemma 4.24: Let M and N be a pseudo manifold of class $\tilde{C}^\infty$, and let $F: M \to N$ be a any map. If $\{(u_i, \theta_i)_{i\in I}\}$, $\{(v_j, \vartheta_j)_{j\in J}\}$ are $\tilde{C}^\infty$atlases for a two pseudo manifolds M, N respectively, and if for each i, j , $\vartheta_j \circ F \circ {\theta_i}^{-1}$ is smooth on its domain of definition, then F is smooth.

Proof: Let $p \in M$, and let a two $\tilde{C}^\infty$charts (u_i, θ_i) , $(v_j, \vartheta_j)$from the given atlases, such that $p \in u_i$ and $F(p) \in v_j$. From smoothness of $\vartheta_j \circ F \circ {\theta_i}^{-1}$, the set $u = F^{-1}(v_j) \cap u_i$ is open set in M, and $F(u) \subset v_j$. Hence the $\tilde{C}^\infty$charts $(u_i, \theta_i|_u)$, (v_j, ϑ_j) satisfy the conditions required in the definition of smoothness.

Proposition 4.25: If M_2 is a pseudo manifold of class $\tilde{C}^\infty$and $\rho: M_1 \to M_2$ is any pseudo topological covering map, then M_1 has a smooth pseudo manifold structure such that F is a smooth covering map.

Proof: Let $p, q \in M_1$` such that, $p \neq q$ . Suppose $\tau(m) \cap \tau(n) \neq \emptyset$ then there exists a filter $\mathfrak{f} \in \tau(m) \cap \tau(n)$ , hence $\mathfrak{f} \downarrow_m$ and $\mathfrak{f} \downarrow_n$ on $M_1$`. Therefore $\rho(\mathfrak{f}) \downarrow_{\rho(m)}$ and $\rho(\mathfrak{f}) \downarrow_{\rho(n)}$, and since M_2 is separated then, $\rho(m) = \rho(n)$. But ρ is injective then it is contradiction. Hence, $\tau(m) \cap \tau(n) = \emptyset$. Thus M_1` is separated. Any point $p \in M_2$ has an evenly covered neighborhood u. Shrinking u if necessary, we may assume also that it is the domain of a coordinate map $\theta: u \to \Xi$. Letting v be a component of $\rho^{-1}(u)$ and $\tilde{\theta} = \theta \circ \rho: \tilde{u} \to \Xi$, it is clear that $(\tilde{u}, \tilde{\theta})$ is a chart on M_2. If two such charts $(\tilde{u}, \tilde{\theta})$ and $(\tilde{v}, \tilde{\varphi})$. Overlap, the transition map can be written $\tilde{\varphi} \circ \tilde{\theta}^{-1} = (\varphi \circ \rho/_{\tilde{u}\cap\tilde{v}}) \circ (\theta \circ \rho/_{\tilde{u}\cap\tilde{v}})^{-1} = \varphi \circ \rho/_{\tilde{u}\cap\tilde{v}} \circ (\rho/_{\tilde{u}\cap\tilde{v}})^{-1} \circ \theta^{-1} = \varphi \circ \theta^{-1}$. Which is smooth. Thus, the collection of all such charts defines a smooth structure on M_1.

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