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# Absolute Difference Energy for Color Graph and Its Bounds

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**Abstract**: A proper vertex coloring of a graph *G* is an assignment of colors to the vertices so that the adjacent vertices get different colors. The chromatic number $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of G. In this paper, bounds for the absolute difference between the color signless Laplacian energy and the color Laplacian energy are obtained. Further, a comparative study of the two bounds obtained is carried out.

**Keywords:** Colored graph, absolute difference energy, non-regular graph, star graph.

**Mathematics Subject Classification**: 05C15, 05C50.

## 1. Introduction

The concept of color Laplacian energy and color signless Laplacian energy of a graph was introduced by Pradeep G. Bhat and Sabitha D' Souza in [4] and [5]**.** Spectral Graph Theory is the study of relationship between graph properties and the spectrum of the adjacency matrix or Laplacian matrix or signless Laplacian matrix. The concept of energy originates from Chemistry to approximate the total $\pi$-electron energy of a molecule. In the field of Spectral Graph Theory, the eigenvalues of several different matrices such as adjacency matrix, Laplacian matrix, distance matrix, *L*-matrix has been studied effectively through more than five decades. Let *G* be a simple undirected graph with vertex set $|V(G)| = n$ and edge set $|E(G)| = m$. An adjacency matrix is the $n \times n$ matrix $A(G) = [a_{ij}]$ in which $a_{ij}$ is the number of edges joining the vertices $v_i$ *and* $v_j$. The eigenvalues of the adjacency matrix of a graph *G* are called as the spectrum of a graph *G*. The various characteristics of the spectrum for different types of graphs can be seen in [9]. The energy of a graph is defined as $E(G) = \sum_{i=1}^{n}|\lambda_i|$, where $\lambda_i's$ are the eigenvalues of a graph *G*. Many of the properties of graph energy are studied in [3], [9], [11], [13] and [15]. This idea was further extended by the introduction of the Laplacian [12] and signless Laplacian energies defined as $LE(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$ and $LE^{+}(G) = \sum_{i=1}^{n}\left|\mu_i{}^{+} - \frac{2m}{n}\right|$ where $\mu_i$ and $\mu_i{}^{+}$ are the Laplacian and signless Laplacian eigenvalues of a graph *G* respectively where *m* is the number of edges and *n* is the number of vertices in *G*. The properties of these matrices in regard to their spectrum and the comparisons with respect to their energy are found in [1], [6], [7], [8], [9], [12], [16], [17] and [18]. An important property of the Laplacian matrix and the signless Laplacian matrix is that they are positive semi definite. The color energy of a graph *G* was introduced in [3] and is given by $E_{\chi}(G) = \sum_{i=1}^{n}|\lambda_i|$, where the $\lambda_i's$ are the color eigenvalues of a graph *G*. The color Laplacian energy of a graph *G* is defined as $LE_{\chi}(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$, and in a similar way, the color signless Laplacian energy of a graph *G* is defined as $LE^{+}{}_{\chi}(G) = \sum_{i=1}^{n}\left|\mu_i{}^{+} - \frac{2m}{n}\right|$, where $\mu_i$ and $\mu_i{}^{+}$ are the color Laplacian and color signless Laplacian eigenvalues of a graph *G* respectively with matrices $L_{\chi}(G) = D(G) - A_{\chi}(G)$ and $L^{+}{}_{\chi}(G) = D(G) + A_{\chi}(G)$ respectively. The color Laplacian and color signless Laplacian energy are same when the graph is regular. In general, color signless Laplacian energy is more than the color Laplacian energy but this is not true for few graphs. Hence, finding absolute difference energy of a colored graph is more significant. Also, the color energy will be greater when we use minimum color.

In this paper, some upper bounds for the absolute difference between the color Laplacian and color signless Laplacian energies of graphs in terms of minimum colors. Also, a comparison is made between the two computed upper bounds for fewer colored graphs. It is observed that the computed sharp upper bound is better than the other upper bound.

## 2. Preliminaries

In this section, some basic definitions and theorems which are needed to prove the main results are presented.

Let *G* be a colored graph and if $c(v_i)$ is the color of vertex $v_i$ and the color matrix $A_c(G) = [a_{ij}]$ of *G* is a square matrix of order *n* whose entries are:

$$[a_{ij}] = \begin{cases} 1 & if\ \ v_i\ \ and\ \ v_j\ \ are\ \ adjacent \\ & with\ \ c(v_i) \neq c(v_j) \\ -1 & if\ \ v_i\ \ and\ \ v_j\ \ are\ \ non\ \ adjacent \\ & with\ \ c(v_i) = c(v_j) \\ 0 & otherwise \end{cases}$$

The eigenvalues of $A_c$(G) are called color eigenvalues. The energy of a graph with respect to a given coloring is the sum of the absolute values of the color eigenvalues of *G* and it is called as energy of colored graph or color energy of a graph *G*.

**Definition 2.1** [10]. A star is a tree consisting of one vertex adjacent to all others. The *n*-vertex star is the biclique $K_{1,n-1}$.

**Definition 2.2** [1]. The null graph is denoted as $\overline{K_n}$, where $K_n$ is the complete graph on *n* vertices.

**Theorem 2.3** [4].

If $K_{1,n-1}$ is a star graph of order *n*, then

$$L_{spec_\chi}(K_{1,n-1}) = \begin{Bmatrix} 0 & (n-1)+\sqrt{n-1} & (n-1)-\sqrt{n-1} \\ n-2 & 1 & 1 \end{Bmatrix}$$

and $LE_\chi(K_{1,n-1}) = \frac{4(n-1)(n-2)}{n}$.

**Theorem 2.4** [5].

If $K_{1,n-1}$ is a star graph of order *n*, then

$$L_{spec_\chi}{}^{+}(K_{1,n-1}) = \begin{Bmatrix} 2 & 1+\sqrt{n^2-3n+3} & 1-\sqrt{n^2-3n+3} \\ n-2 & 1 & 1 \end{Bmatrix}$$

and $LE_\chi{}^{+}(K_{1,n-1}) = \frac{2}{n}\left[n-2+n\sqrt{n^2-3n+3}\ \right]$.

**Theorem 2.5** [2].

If *G* is a null graph of order *n*, then its color energy

$$E_\chi(G) = 2(n-1).$$

**Theorem 2.6**[1].

Let $A, B \in \mathbb{R}^{n\times n}$. Then $C = A + B$ and

$$E(C) \leq E(A) + E(B).$$

Moreover, equality holds if and only if there exists an orthogonal matrix *P* such that *PA* and *PB* are both positive semi-definite matrices.

## 3. Main Results

This section investigates the color signless Laplacian energy and Laplacian energy of a colored graph *G*. Two main results which give the upper bound for the absolute difference between the energies of the signless Laplacian and the Laplacian color graphs in terms of the chromatic number of the given graph. The following theorem is analogous to that of theorem 2.6 but it holds good for the color graph matrices and color energies.

**Theorem 3.1**

Let $A, B \in \mathbb{R}^{n\times n}$. Let $C = A + B$. Then

$$E_\chi(C) \leq E_\chi(A) + E_\chi(B) \qquad (1)$$

**Theorem 3.2**

Let *G* be colored, non-regular $(n, m)$ −graph. Then

$$\left|LE_\chi{}^{+}(G) - LE_\chi(G)\right| \leq 2E_\chi(G) \qquad (2)$$

The equality holds if *G* is a null graph of order 1.

**Proof:**

From the equalities

$$L_\chi(G) - \frac{2m}{n} I_n = D(G) - \frac{2m}{n} I_n - A_\chi(G)$$

$$L^{+}{}_{\chi}(G) - \frac{2m}{n} I_n = \ D(G) - \frac{2m}{n} I_n - A_{\chi}(G)$$

it follows that

$$\left(L^{+}{}_{\chi}(G) - \frac{2m}{n} I_n\right) - \left(L_{\chi}(G) - \frac{2m}{n} I_n\right) = \ 2A_{\chi}(G)$$

Then, $L_{\chi}(G) - \frac{2m}{n} I_n = -2A_{\chi}(G) + \ L^{+}{}_{\chi}(G) - \frac{2m}{n} I_n$

and $L^{+}{}_{\chi}(G) - \frac{2m}{n} I_n = 2A_{\chi}(G) + L_{\chi}(G) - \frac{2m}{n} I_n$

by theorem 3.1, we obtain

$$LE_{\chi}(G) = E\left(L_{\chi}(G) - \frac{2m}{n} I_n\right) \leq E(-2A_{\chi}(G)) + E\left(L^{+}{}_{\chi}(G) - \frac{2m}{n} I_n\right) = 2E_{\chi}(G) + \ LE_{\chi}{}^{+}(G)$$

Similarly,

$$LE_{\chi}{}^{+}(G) = E\left(L^{+}{}_{\chi}(G) - \frac{2m}{n} I_n\right) \leq E(2A_{\chi}(G)) + E\left(L_{\chi}(G) - \frac{2m}{n} I_n\right) = 2E_{\chi}(G) + LE_{\chi}(G)$$

and thus equation (1) follows.

Finally, if *G* is the null graph of order 1, then it is immediate that the inequality (1) holds as equality.

**Theorem 3.3**

Let *G* be a colored, non-regular $(n, m)$ −graph with $n \geq 5$

$$\left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| \leq \ \frac{4}{n}(n^2 - 3n + 3) - 2\left(1 + \sqrt{(n^2 - 3n + 3)}\right)$$

and equality holds if *G* is a star graph.

**Proof:**

By theorem 3.2, $\left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| \leq \ 2E_{\chi}(G)$

and equality holds if *G* is a null graph of order 1.

From theorem 2.5, we have for a null graph

$$E_{\chi}(G) \ = \ 2(n-1) \qquad (3)$$

Using (1) and (2) we obtain,

$\left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| \leq \ 2 \times 2(n-1) = 4(n-1)$ (4)

Now,

$\frac{4}{n}(n^2 - 3n + 3) - 2\left(1 + \sqrt{(n^2 - 3n + 3)}\right) \leq 4(n-1)$ (5)

using (4) and (5) we get,

$$\left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| - \frac{4}{n}(n^2 - 3n + 3) - 2\left(1 + \sqrt{(n^2 - 3n + 3)}\right) \leq 0$$

$$\Rightarrow \left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| \quad \leq \ \frac{4}{n}(n^2 - 3n + 3) - 2\left(1 + \sqrt{(n^2 - 3n + 3)}\right)$$

and the fact that equality holds if *G* is a star graph follows as a consequence of theorems 2.3 and 2.4.

**Corollary**

1. For $n = 1, 2$ the result is obvious,

$$\left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| = 0$$

2. For $n = 3$,

$$\left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| \leq \left| \ \frac{4}{n}(n^2 - 3n + 3) - 2\left(1 + \sqrt{(n^2 - 3n + 3)}\right)\right|$$

3. For $n = 4$ ,

$$\left|LE_{\chi}{}^{+}(G) - LE_{\chi}(G)\right| < \frac{4}{n}(n^2 - 3n + 3) - 2\left(1 + \sqrt{(n^2 - 3n + 3)}\right)$$

## 4. Comparison between the Bounds

In this section, the bounds given by theorems 3.2 and 3.3 for various colored graph components such as path ($P_n$), complete bipartite ($K_{m,n}$), star ($K_{1,n-1}$) and wheel ($W_n$) graphs are compared and hence conclude that the second bound is the sharper than the first one.

| S.No | Graphs | $\left\| LE_\chi^{+}(G) - LE_\chi(G) \right\|$ | $2E_\chi(G)$ | $\frac{4}{n}(n^2-3n+3)-2\left(1+\sqrt{(n^2-3n+3)}\right)$ |
|---|---|---|---|---|
| 1. | $P_5$ | 0.4194 | 12.9444 | 1.1888897449 |
| 2. | $P_6$ | 0.4191 | 17.3264 | 2.83484861 |
| 3. | $P_7$ | 0.247 | 22.3144 | 4.577871274 |
| 4. | $P_8$ | 0.7644 | 26.7138 | 6.385122951 |
| 5. | $K_{2,3}$ | 0.0074 | 16 | 1.1888897449 |
| 6. | $K_{3,4}$ | 0.0003 | 24 | 4.577871274 |
| 7. | $K_{3,5}$ | 1.0282 | 48 | 6.385122951 |
| 8. | $K_{2,4}$ | 0.9286 | 20 | 2.83484861 |
| 9. | $K_{1,4}$ | 1.1888 | 16 | 1.1888897449 |
| 10. | $K_{1,5}$ | 2.8349 | 20 | 2.83484861 |
| 11. | $K_{1,6}$ | 4.5788 | 24 | 4.577871274 |
| 12. | $K_{1,7}$ | 6.3852 | 48 | 6.385122951 |
| 13. | $W_5$ | 0.8722 | 18.2464 | 1.1888897449 |
| 14. | $W_6$ | 0.9665 | 22.6732 | 2.83484861 |
| 15. | $W_7$ | 0.1758 | 28 | 4.577871274 |
| 16. | $W_8$ | 1.833 | 30.9062 | 6.385122951 |
| 17. | $W_9$ | 0.3588 | 36.8028 | 8.231131129 |

## 5. Conclusion

This paper presents two different upper bounds for the absolute difference between the color Laplacian and color signless Laplacian energies of colored graphs. A comparison is done for the two kinds of bounds for various graphs depicts that the second bound given by theorem 3.3 is better than the first bound given by theorem 3.2. The study can be further improved to find better sharper bounds and also to establish relation between those bounds.

## 6. References


[1] N. Abreu, D.M. Cardoso, I. Gutman, E.A. Martins, M. Robbiano, Bounds for the signless Laplacian energy, Linear Algebra Appl. 435 (2011) 2365-2374.

[2] C. Adiga, E. Sampathkumar, M.A. Sriraj, A.S. Shrikanth, Color energy of a graph, Proc. Jangjeon Math. Soc. 16 (2013) 335-351.

[3] R. Balakrishnan, The energy of a graph, Linear algebra Appl. 387(2004) 287-295.

[4] P.G. Bhat and S. D' Souza., Color Laplacian Energy of a Graph, Proc. Jangjeon Math. Soc. 18 (2015) 321-330.

[5] P.G. Bhat and S. D' Souza., Color signless Laplacian energy of graphs, AKCE International Journal of Graphs and Combinatorics 14(2017) 142-148.

[6] D. Cokilavany, Extended Energy of Some Special Class of Unicyclic Graphs, Jour of Adv Research in Dynamical & Control Systems, 12(4) (2020) 460-467.

[7] D. Cokilavany, On Some Bounds of Extended Signless Laplacian Matrix Indices, Jour of Adv Research in Dynamical & Control Systems, 12(4) (2020) 468-473.

[8] D. Cokilavany, Extended energy of some standard graphs, Malaya Journal of Matematik, 8(2) (2020) 510-516.

[9] D. Cvetkovic, M. Doob, H. Sachs, Spectra of Graphs – Theory and Application, Barth, Heidelberg, 1995.

[10] Douglas B. West, Introduction to Graph theory, 2nd ed., 2009.

[11] I. Gutman, The energy of a graph, Ber. Math. Statist. Sekt. Forschungsz. Grazer 103(1978) 1-22.

[12] I. Gutman, B. Zhou, Laplacian energy of a graph, Lin. Algebra Appl. 414 (2006) 378-382.
[13] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann(Eds), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, 196-211.
[14] J. A Bondy and U. S. R Murty, Graph theory with Applications, 1982.
[15] G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graphs, MATCH Commun. Math. Comput. Chem. 60 (2008) 461-472
[16] B. Zhou and Ivan Gutman, On Laplacian energy of graphs, MATCH Commun. Math. Comput. Chem. 57(2007) 211-220.
[17] B. Zhou, New upper bounds for Laplacian energy, MATCH Commun. Math. Comput. Chem. 62(2009) 553-560.
[18] B. Zhou, More on energy and Laplacian energy, MATCH Commun. Math. Comput. Chem. 64(2010) 75-84.