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A Study on Kepler's law of Nonlinear Differential Equation with the Solution of Lambert's Problem in a Three-Dimensional Intercept Space

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Abstract: In this paper, we have discussed about three-dimensional intercept problem. We have also applied nonlinear differential equations, Lagrangian coefficient and Kepler's law for discussing the three-dimensional intercept problem. We have also discussed about Lambert's problem for the study of minimum energy orbit by using Lagrangian coefficients.

Keywords: Optimization, Minimum orbit energy, Time of Flight, Lagrangian Coefficients and Lambert's problem.

I. Introduction:

According to Albert Einstein, the determination of the true movement of the planets, including the earth, this statement also known as Kepler's first great problem. Another second problem is that What are the mathematical laws controlling these movements? Clearly it is said that if it were possible for the human spirit to accomplish it, presupposed the solution of the first. Kepler explained on Tycho's observations of Mars and expressed it with some simple geometrical theory of motion. He had made three revolutionary assumptions: (a) that the orbit was a circle with the sun slightly off-centre, (b) that the orbital motion took place in a plane which was fixed in space, and (c) that Mars did not necessarily move with uniform velocity along this circle. Firstly, Kepler's first task was to find out the radius of the circle and the direction of the axis connecting perihelion and aphelion. At the end, he accomplished his target by representing within 2 arc-minutes the position of Mars at all 10 oppositions recorded by Tycho. It is to Kepler's constant approval that he established the basis for a complete reformation of astronomy. His outcome proved that the earth did not move with uniform speed, but faster or slower according to its distance from the sun. We will then turn our attention to the solution of what has come to be known as "the Kepler problem"-predicting the future position and velocity of an orbiting object as a function of some known initial position. and velocity and the time-of-flight [5]. The significance of Lambert's theorem by using two-point boundary value problem (TPBVP), states that "the orbital transfer time depends only upon the semi-major axis, the sum of the distances of the initial and final points of the arc from the centre of force, and the length of the chord joining these points [4]. The two-point boundary value problem for Keplerian motion, also known as Lambert's problem, is a classical one in astrodynamics. The classical orbit intercept applications are often formulated and solved as Lambert-type problems, where the time-of-flight (TOF) is specified. In three-dimensional intercept problems, we know that choosing a relevant TOF is repeatedly a difficult one and an iterative process. In this work, we begin with a standard derivation of Kepler's equation and describe how it set up to a new mathematical model. This work overcomes the limitation of classical Lambert's problem by reformulating the intercept problem in terms of a minimum-energy application, which then give rise to both initial interceptor velocity and the TOF for the minimum energy transfer. The energy minimum form of Lambert's problem is solved by introducing the classical Lagrangian coefficients and universal variable in the problem necessary conditions. The optimization problem is introduced by using the classical Lagrangian f and g coefficients, which map initial position and velocity vectors to future times, and a universal time variable x. This optimization problem generates a generalized formulation problem for minimizing the TOF [1]. At present, there is no other analytical approach apart from the geometrical analysis for solving the problem. In this work, we proposed an analytical method for solving the Lambert's minimum energy problem. It is to note that we obtained the minimum initial velocity by applying the nonlinear constrained optimization problems which is similar to the determined minimum energy orbit. The most interesting fact is that using alternative technique could help us to gain new insight for solving many orbital problems [2].

II. Mathematical Modeling:

In this mathematical modeling, we have introduced Kepler's equation that describes many geometric properties of the orbit of a body which is generally subject to a central force for spacecraft. We will consider this equation for determining the time of flight (TOF) because it associates the time of flight from periapsis to eccentric anomaly, semimajor axis and eccentricity. This equation has played a crucial role in the history of both physics and mathematics mainly in classical celestial mechanics. Kepler then introduce the notation for the mean anomaly M as,

$M = E - esinE = \sqrt{\frac{\mu}{a^3}}\ (t - T)$ ………..((1)

We considered time of flight as a function of eccentric anomaly as a new term introduced by Kepler in connection with elliptical orbits.

$$t - T = \sqrt{\frac{a^3}{\mu}}(E - esinE)$$

Where E=eccentric anomaly, e=eccentricity, μ= gravitational constant, a = semimajor axis of orbit, T= time of periapsis passage and t= time of flight respectively.

We also introduced the mean motion (n) through Kepler's third law which relates p to the semimajor axis, so n is a function of a.

$$n = \frac{2\pi}{p} = \frac{\mu}{\sqrt{a^3}}$$

Practically, it is possible to obtain time-of-flight equations analytically by applying the dynamical equation of motion and integral calculus. We derivation in which the eccentric anomaly arises quite naturally in the course of the geometrical arguments. This derivation is expressed more for its historical value than for actual use. The universal variable approach is firmly suggested as the best method for general use.

$$TOF = \pi\sqrt{\frac{a^3}{\mu}}$$

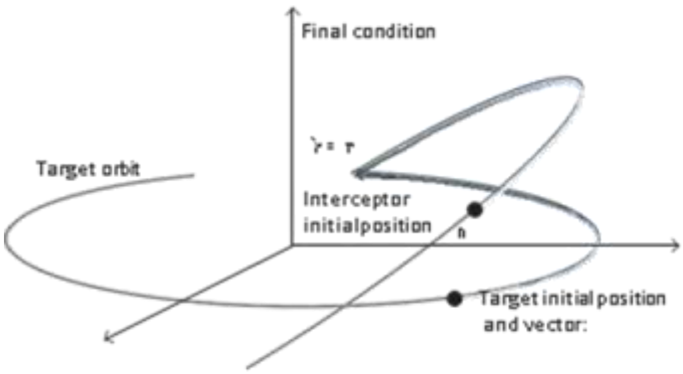

Fig.1 Geometry of the minimum-energy problem for a TOF

From eqn. (1), we can rewrite the mean anomaly as,

$M = n\,(t - T) = E - esinE$,

which is often referred to as Kepler's equation.

We know that the loss of numerical accuracy can occur near $e \backsim 1$, in which the time of flight involving the eccentric anomalies (E) does not work out properly near parabolic orbits in the Kepler's equation because the trial-and-error solutions are converge too slowly or not at all. This great loss of computational accuracy near $e = 1$ and the inconvenience of having a different type of equations for each conic orbit can be overcome by introducing the universal variable, x. Moreover, the introduction of this new independent universal variable, x enable us to generate a single time of flight which is reliable for all conic orbits. Thus, the derivative of the universal variable x is defined as,

$\dot{x} = \frac{\sqrt{\mu}}{r}$……….(2)

where, r is the position of spacecraft.

We expressed Kepler's equation in terms of the radial spacecraft coordinate in the following form as

$\sqrt{\mu}t = a\left[x - \sqrt{a}sin\left(\frac{x}{\sqrt{a}}\right)\right] + a\frac{r_0.v_0}{\sqrt{\mu}}\left[1 - cos\left(\frac{x}{\sqrt{a}}\right)\right] + r_0\sqrt{a}sin\left(\frac{x}{\sqrt{a}}\right)$,…(3)

$r = a + a\left[\frac{r_0.v_0}{\sqrt{\mu a}}sin\left(\frac{x}{\sqrt{a}}\right) + \left(\frac{r_0}{a} - 1\right)cos\left(\frac{x}{\sqrt{a}}\right)\right]$…….(4)

where r_0 and v_0 are the initial position and velocity vectors of spacecraft and T is known to be zero without loss of generality respectively. The necessary conditions of r_0 and v_0 describe the position and velocity of an orbiting object as a function of time. The position of the spacecraft can be assessed at a time when the value of the universal variable from (3) is well known. We can obtain the value of x successfully by using Newton's iteration technique when the value of t, time of flight is given.

The f and g expressions:

The f and g functions are referred to as the Lagrange coefficients after Joseph Louis Lagrange (1736–1813), a French mathematical physicist, whose numerous contributions include calculations of planetary motion. We considered four vectors of all coplanar $r_0, v_0, r,$ and v, which are governed by Keplerian motion by assuming that there are no external forces. Therefore, the position and velocity vectors of spacecraft at time t are introduced here in order to calculate v and r in terms of v_0, r_0 and x as,

$r = fr_0 + gv_0$

$v = \dot{f}r_0 + \dot{g}v_0$……………… (5)

It is seen that the position and velocity vectors r and v are indeed linear combinations of the initial position and velocity vectors. The Lagrange coefficients and their time derivatives in these expressions are themselves functions of time and the initial conditions where $f, g, \dot{f}$ and $\dot{g}$ are scalar time-dependent constants which are subject to the following constraint:

$f\dot{g} - \dot{f}g = 1$ (conservation of angular momentum) …………………………..(6)

where $f = 1 - \frac{a}{r_0}\left[1 - cos\left(\frac{x}{\sqrt{a}}\right)\right]$,

$g = t - \frac{a}{\sqrt{\mu}}\left[x - \sqrt{a}\, sin\left(\frac{x}{\sqrt{a}}\right)\right]$………..(7)

$$\dot{f} = -\frac{\sqrt{\mu a}}{rr_0}\, sin\left(\frac{x}{\sqrt{a}}\right)$$

$\dot{g} = 1 - \frac{a}{r}\left[1 - cos\left(\frac{x}{\sqrt{a}}\right)\right]$……(8)

We introduced the Lagrangian function in order to find the maximum or minimum of a function $\underline{f(x)}$ subjected to the equality constraint $\underline{g(x)}$ =0 , which have to be satisfied exactly by the choose values of the variable as follows:

$\mathcal{L}(x,\lambda) = f(x) - \lambda g(x)$ …….(9)

Where, this function $\mathcal{L}$ is known as Lagrangian, which is a function of x and a new variable λ is referred to as a Lagrange multiplier. From eqn.(7), we can re-write eqn.(9) as,

$$\mathcal{L}(x,\lambda) = 1 - \frac{a}{r_0}\left[1 - cos\left(\frac{x}{\sqrt{a}}\right)\right] - \lambda\left[t - \frac{a}{\sqrt{\mu}}\left\{x - \sqrt{a}\, sin\left(\frac{x}{\sqrt{a}}\right)\right\}\right]$$

$$= 1 - \frac{a}{r_0} + \frac{a}{r_0}cos\left(\frac{x}{\sqrt{a}}\right) - \lambda t - \frac{a\lambda}{\sqrt{\mu}}x + \frac{\lambda a\sqrt{a}}{\sqrt{\mu}}\, sin\frac{x}{\sqrt{a}}$$

$= 1 - \lambda t - a\left(\frac{1}{r_0} + \lambda\frac{x}{\sqrt{\mu}}\right) + a\left[\frac{1}{r_0}\, cos\frac{x}{\sqrt{a}} + \lambda\sqrt{\frac{a}{\mu}}\sin\frac{x}{\sqrt{a}}\right]$ ……………….(10)

The basic idea of this method is to remodel a constrained problem into a set such that the derivative test of an unconstrained problem can still be obtained. The great advantage of this method is that we can obtained the optimization without certain parameterization in terms of the constraints. Most importantly, the method of Lagrange multiplier is widely used to solve challenging constrained optimization problems. The energy minimum form of Lambert's problem is solved by introducing the classical Lagrangian coefficients and universal variable in the necessary condition.

III. Nonlinear optimization approach of Lambert's problem:

Another expressions for f and g is formulated for Lambert's problem by using the following relationship between two position vectors and one initial-velocity vector v_0 as

$r_1 = fr_0 + gv_0$ ………..(11)

where f and g are the two time-independent variables and designated as follows

$f = 1 - \frac{r_1}{p}(1 - cos\Delta v)$ …(12)

$g = \frac{r_1 r_0 sin\Delta v}{\sqrt{\mu p}}$ ……………...(13), where p is the semi-parameter.

Now, the Lambert's problem is reformulated based on the constrained optimization method by selecting $x = [v_0^T p]^T$ which is given as,

$fr_0 + gv_0 - r_1 = 0$………………..(14)

At this time, the interceptor orbit energy for the magnitudes of the position and velocity vectors has been set out because it's valid for all orbits which is defined as,

$\xi = \frac{v_0^2}{2} - \frac{\mu}{r_0}$ ……………………….(15)

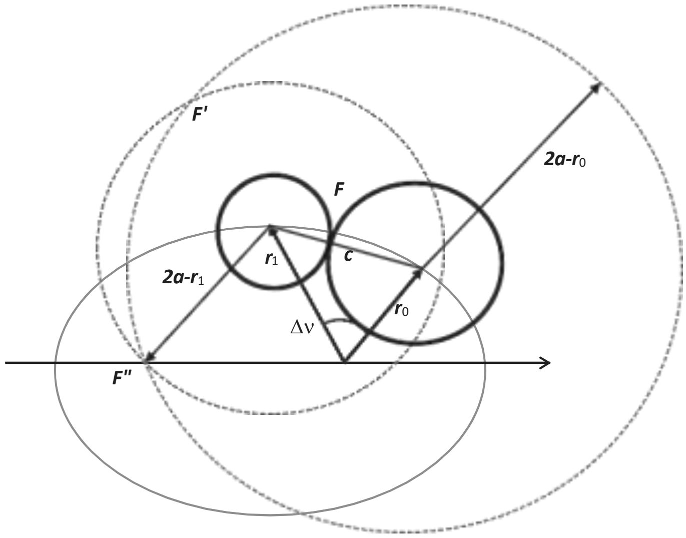

Fig:2 Geometry for the Lambert problem.

It is to be noted that the second term of the orbit energy is constant and known, because of that we can establish the performance index simply without loss of generality as

$\mathcal{J}(x) = \frac{1}{2} v_0^T v_0$……………………….(16)

The Hamiltonian is established by adding the constraint of an undetermined Lagrange-multiplier vector as follows,

$H = \frac{1}{2} v_0^T v_0 + \lambda^T (f r_0 + g v_0 - r_1)$ …………………………………..(17)

where $\lambda \epsilon \mathcal{R}^3$ is the Lagrange-multiplier vector.

The necessary conditions have been provided inorder to minimize the performance index with respect to the augmented variables as given below,

$\frac{\partial L}{\partial v_0} = v_0^T + g\lambda^T = 0$ ……..(18)

$\frac{\partial L}{\partial p} = \lambda^T \left(\frac{r_1}{p^2} (1 - cos\Delta v) r_0 - \frac{r_1 r_0 \, sin(\Delta v)}{2\sqrt{\mu p^3}} v_0 \right)$……………...(19)

We can obtained the following equations from eqn.(12) and eqn.(13) as

$\frac{r_1}{p^2} (1 - cos\Delta v) = \frac{1-f}{p}$…….(20)

$\frac{r_1 r_0 \, sin(\Delta v)}{2\sqrt{\mu p^3}} = \frac{g}{2p}$……………(21)

Then, the necessary condition of eqn.(19) is classified by substituting the above two equations in eqn.(19)

$\lambda^T \left(\frac{1-f}{p} r_0 - \frac{g}{2p} v_0 \right) = 0$…..(22)

From eqn.(10), we get the following results of Lagrange-multiplier vector and the initial velocity vector respectively,

$\lambda^T = -\frac{1}{g} v_0^T$…………….(23)

$v_0 = \frac{r_1 - f r_0}{g}$……………...(24)

By multiplying pg on both sides of eqn.(22) and substituting the above two equations, we obtained the results as,

$(r_1^T - f r_0^T) \left(r_0 - f r_0 - \frac{1}{2} (r_1 - f r_0) \right) = 0$………………………...(25)

Henceforth, we can derive the second order equation with respect to f by employing the above equation as shown below: -

$f^2 - 2f + \frac{2 r_0^T r_1 - r_1^2}{r_0^2} = 0$…(26)

In order to obtained the solution of this equation in a simpler way is described as,

$f = 1 \pm \sqrt{1 + \frac{r_1^2 - 2 r_0^T r_1}{r_0^2}}$……………..(27)

For the case of eccentricity as $f \leq 1$, we choose the value of as

$$f = 1 - \frac{\sqrt{r_0^2 + r_1^2 - 2r_0^T r_1}}{r_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

We get the result of semi-parameter by comparing eqn.(12) with eqn.(28) as shown below:-

$$\frac{r_1}{p}(1 - cos\Delta v) = \frac{\sqrt{r_0^2 + r_1^2 - 2r_0^T r_1}}{r_0} \quad \ldots\ldots..(29)$$

Now, by rearranging the above equation, we get the result of minimum energy semi-parameter, p as

$$p_{min} = \frac{r_0 r_1}{\sqrt{r_0^2 + r_1^2 - 2r_0^T r_1}}(1 - cos\Delta v) \quad \ldots.(30)$$

It is prominent from the above equation that we can apply the considered expressions of f and g from eqn.(12) and eqn.(13) because the above semi-parameter attained by the optimization method is similar to the earlier discussed geometrical minimum solution in eqn.(12), such that the minimum initial velocity gives as

$$v_0 = \frac{\sqrt{\mu p_{min}}}{r_0 r_1 sin(\Delta v)}\left[r_1 - \left(1 - \frac{r_1}{p_{min}}(1 - cos\Delta v)\right) r_0\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(31)$$

IV. Conclusions:

This study proposed on Kepler's equation used to explain the TOF equation followed by a description of the universal variable used for the problem formulation. The mathematical advantage of this approach is that the problem has a unique optimal solution, rather than the family of solutions that characterize the classical Lambert's problem. Mathematically, the problem is defined by a constrained optimization algorithm. This constrained optimization algorithm is introduced in formulating the problem for operating the near-parabolic orbits that appears in intercept applications. Analytically, this is handled in a very comprehensive way by introducing a universal variable that allow a single TOF equation to be established that is reliable for all conic orbits. It is obviously understood that the applicability of Lambert's problem to astrodynamics is somehow an efficient solution algorithm which can be extremely useful in almost any part of space mission design and operation, from orbit determination to trajectory optimization. In this work, all the approaches are studied based on a numerical procedure where the value of the free parameter is searched iteratively. It is also enabled to state the problem in terms of universal variables, so that a uniform parametrization is determined for all type of transfers along arcs of elliptical, parabolic, or hyperbolic orbits. The proposed algorithm is predicted to be broadly practical for all classes types of intercept problem that have a Lambert-like character. By solving the nonlinear 3D, intercept problem through minimizing energy, the TOF and initial velocity is calculated. Then, applying minimum-energy velocity obtained for the interceptor shows that the final distance between the two orbits is zero at the computed TOF. It is observed that the problems formulated by the universal variable and f and g expressions in this paper are explained in 3D space for sustaining the design of arbitrary intercept problems with minimum energy. In general terms, algorithm is introduced for generalizing the classical Lambert's transfer problem, where the determination of time of-flight (TOF) for a spacecraft intercept, in arbitrary three-dimensional orbit. The optimization problems that are solved to find the transfer trajectories are divided into direct or indirect methods. Direct method requires fairly fewer function computations whereas Indirect methods may exhibit rapid convergence when compared to a direct method. In this study, we applied constrained optimization method to solve the Lambert minimum-energy problem. Eventually, it was shown that the result from the proposed approach coordinated with the solution of the geometrical approach. The resulting minimum-energy intercept solution algorithm has important applications on spanning, rendezvous, targeting, interplanetary trajectory design, and so on.

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