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# Study of New Type of Vector Fields on Light like Hypersurfaces of Lorentzian Manifolds

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**Abstract –**In this paper we study new type of vector fields lying on lightlike hypersurfaces of a Lorentzian manifold$(\bar{\boldsymbol{M}}, \boldsymbol{g})$. We obtainedsome results dealing with new type of vector fields and discussed for totally umbilical lightlike hypersurfaces and totally geodesic lightlike hypersurfaces. Furthermore, Ricci soliton lightlike hypersurfaces admitting new type of vector fields are studied and some characterizations for this frame of hypersurfaces are obtained.

**Keywords:** New type of vectorfield;lightlikehypersurface;Lorentzianmanifold;Riccisoliton**.**

## 1. Introduction

In 1943, K. Yano [25] proved that there exists a smooth vector field $\boldsymbol{v}$, so called concurrent on a Riemannian manifold $(\bar{\boldsymbol{M}}, \boldsymbol{g})$ which satisfies the following condition. For every vector field $\boldsymbol{F}$ tangent to$\bar{\boldsymbol{M}}$

$$\boldsymbol{D_F V = F}$$

Where$D$is the Levi-Civita connection with respect to Riemannian metric $\bar{g}$on $\overline{\mathrm{M}}$. Applications of concurrent vector fields have been investigated and Riemannian andSemi-Riemannian manifolds equipped with concurrent vector fields have been intensely studied by various authors ([6], [8]).

Beside these facts, the notion of a Ricci soliton ([16], [17]), is initially observed by Hamilton's Ricci flow and Ricci solitons drew attention after $G$. Perelman and applied Ricci solitons to solve the Poincare conjecture.

A Riemannian manifold$(\overline{\mathrm{M}}, g)$ with a metric tensor $g$ is called a Ricci soliton [18] if, thereexists a smooth vector field $v$ tangent to $M$ satisfying the following equation

$$\frac{1}{2} L_v g(F,J) + Ric(F,J) = \lambda g(F,J) \tag{1}$$

In the last twenty years, the hypothesis of geometric flows is the most important mathematical tools to describe the geometric structures in Riemannian geometry. A specific section of solutions on which the metric evolves by diffeomorphisms has a significant impact in the investigation of singularities of the flows as they show up as possible singularity models. They are frequently called soliton solutions.

Yamabe flow was presented by Hamilton [16], simultaneously with Ricci flow. The Ricci soliton and Yamabe soliton are special solutions of the Ricci flow and Yamabe flow, respectively. The Ricci soliton and the Yamabe soliton are equivalent for dimension$n = 2$. Although, in dimension $n > 2$, they are not identical and in the course of recent years, the theoretical concept of geometric flows, for example, Ricci flow and Yamabe flow has been the focal point of fascination of numerous geometers. As of late, in 2019, Guler and Crasmareanu [15] presented the investigation of another geometric flow under the name Ricci-Yamabe map. This map is nothing but a scalar combination of Ricci and Yamabe flow.

This is additionally named $(\alpha, \beta)$ type Ricci-Yamabe flow. This type of flow is advancement for the metrics on$(\bar{M}, g)$,the Riemannian manifold defined by [15]

$$\frac{\delta}{\delta t} g(t) = \beta r(t) g(t) - 2\alpha S(t), \qquad g_0 = g(0) \tag{2}$$

Where$S$is the Ricci tensor, $r$ denotes the scalar curvature, and$\alpha, \beta, \gamma \in R$. One can think Ricci-Yamabe flow as Riemannian or singular Riemannian or semi-Riemannian flow because of the signs of $\alpha$and β the involved scalars. This sort of different choices can be valuable in some mathematical or physical models such as relativistic theories.

Therefore, normally Ricci-Yamabe soliton arises as the constraint of the soliton of Ricci-Yamabe flow. Another strong inspiration that started the investigation of Ricci-Yamabe solitons is that, in spite of the fact that Ricci solitons and Yamabe solitons are identical in dimensional 2, in higher dimension they are basically different.

A Ricci-Yamabe soliton [9] on $(\bar{M}, g)$ is a data $(g, F, \lambda, \alpha, \beta)$ fulfilling

(3) $$\mathcal{L}_F g(F,J) = -2\alpha S(F,J) - (2\lambda - \beta r) g(F,J)$$

Where£ being the Lie-derivative, $S$ indicates the Ricci tensor, $r$ denotes the scalar curvature and $\lambda, \alpha, \beta \in R$. If $f$ is a smooth function and $F$ is the gradient of $f$on $N$, then the foregoing notion is named gradient Ricci-Yamabe soliton [9] and the equation (3) transforms to

(4) $$\bar{D}^2 f = -\alpha S(F,J) - (\lambda - \frac{1}{2}\beta r) g(F,J)$$

where the Hessian of$f$is denoted by $\bar{D}^2 f$.

The Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be expanding for$\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$.If $\lambda, \beta$and $\alpha$are smooth functions on $\bar{M}$, then a Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is called an almost Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton).If $\beta = 0, \alpha = 1$, then Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) turns into Ricci soliton (or gradient Ricci soliton) [16]. Similarly, it turns into Yamabe soliton (or gradient Yamabe soliton) [17] if$\beta = 1, \alpha = 0$. Alsoif$\beta = -1$, $\alpha = 1$ then it reduces to an Einstein soliton (or gradient Einstein soliton) [7].

The Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be proper if $\alpha \neq 0$, 1. Ricci solitons and Yamabe solitons have been investigated by several authors.

Duggal-Bejancu introduced lightlike geometry of semi-Riemannian manifolds and is completely different from Riemannian and semi-Riemannian one. to overcome this difficulty arisen due to degenerate metric authors obtained transversal bundle for such hypersurfaces. After [11] researchers across the globe studied lightlike hypersurface of manifolds by following Duggal-Bejancu approach. For degenerate hypersurfaces of manifolds we refer ([9],[10], [11], [12], [13], [14]) any many more references therein.

The main purpose of this paper is to investigate new type of vector fields on lightlikehypersurfaces and Ricci solitons lightlike hypersurfaces of a Lorentzian manifold. However,there are some difficulties to deal with while examining new type of vector fields and Riccisolitons for these kinds of submanifolds.

(7)

(8)

The first problem is that since the inducedmetric is degenerate and hence not invertible for a lightlike hypersurface, some significantdifferential operators such as the gradient, divergence, Laplacian operators with respectto the degenerate metric cannot be defined. To get rid of this problem, we consider theassociated metric defined with the help of a rigging vector field. The second main problem is that the Ricci tensor of any lightlike hypersurface is not symmetric. In this case, the Riccisoliton equation loses its geometric and physical meanings. To get rid of this problem, weinvestigate this equation on lightlike hypersurfaces with the genus zero screen distributionwhose Ricci tensor is symmetric.

## 2. Preliminaries

A vector field $v$on a Riemannian manifold $M$is said to be a new type of vector field [9] if, for any $X$ tangent to $M$, one has

(5) $$D_F v = B(F) v - F$$

Where the 1-form $B$is associated with the vector field $v$ such that $g(F, v) = B(v)$ and $D$denotes the Levi-Civita connection. In literature, such vector fields are also known as geodesic fields since integral curves of such vector fields are geodesics .

On the other hand the notion of new type of vector fields can naturally be extended to semi-Riemannian manifolds. In that line, Chen, in [8], has provedthat a Lorentzian manifold is a generalized Robertson–Walker spacetime if and only if it admits a timelike concircular vector field.

Concircular vector fields are also important in the study of concircular mappings, that is; conformal mappings preserving geodesic circles [26]. In physics, particularly in general relativity, concircular vector fields have interesting applications. For instance, it has been noted that trajectories of timelike concircular fields in the de-Sitter model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis [24].

Now let us recall some of the elementary and important terminologies about the geometry of lightlike (degenerate) hypersurfaces of semi-Riemannian manifolds .

Assume that$(\boldsymbol{M}, \bar{\boldsymbol{g}}, \boldsymbol{S}(\boldsymbol{TM}))$ to be a null hypersurface of $(\bar{\boldsymbol{M}}, \boldsymbol{g})$. Then over $(\boldsymbol{M}, \bar{\boldsymbol{g}}, \boldsymbol{S}(\boldsymbol{TM}))$, there exists $\boldsymbol{tr}(\boldsymbol{TM})$ a rank 1 unique vector bundle in such a way that for any $\boldsymbol{Z}$ of $\boldsymbol{TM}^{\perp}$ on $\boldsymbol{Y} \subset \boldsymbol{M}$, there exists $\boldsymbol{X}$ a unique section of $\boldsymbol{tr}(\boldsymbol{TM})$ on the coordinate neighborhood$\boldsymbol{Y}$known as null transversal vector field of hypersurface $(\boldsymbol{M}, \bar{\boldsymbol{g}}, \boldsymbol{S}(\boldsymbol{TM}))$. Such that

$$\bar{\boldsymbol{g}}(\boldsymbol{Z}, \boldsymbol{X}) = \mathbf{1}, \qquad \bar{\boldsymbol{g}}(\boldsymbol{X}, \boldsymbol{X}) = \bar{\boldsymbol{g}}(\boldsymbol{X}, \boldsymbol{O})) = \mathbf{0}$$

(6)

$\forall\, O \in \Gamma(S(TM_{|M}))$. Then tangent bundle $T\bar{M}$ is decomposed as

$$\boldsymbol{T\bar{M}}_{|\boldsymbol{M}} = \boldsymbol{S}(\boldsymbol{TM}) \oplus (\boldsymbol{TM} \perp \oplus\, \boldsymbol{tr}(\boldsymbol{TM})),$$
$$\boldsymbol{T\bar{M}}_{|\boldsymbol{M}} = \boldsymbol{TM} \oplus \boldsymbol{tr}(\boldsymbol{TM}).$$

Here $tr(TM)$ is known to be as lightlike transversal bundle of hypersurface with respect to $S(TM)$ and $tr(TM)$ is complementary but not orthogonal vector bundle to $TM in \bar{M}_{|M}$[ 9 ].

According to equation (8) for all $F,J \in \Gamma(S(TM_{|M})$, the local Gauss and Weingarten formulas are given as

(9) $\bar{D}_F J = D_F J + B(F,J)$,

$\bar{D}_F X = -A_X F + \tau(F) X$,

(10)

(11) $D_F PJ = D^*{}_F PJ + C(F,PJ)\xi$,

$D_F Z = A^*{}_Z F + \tau(F) Z$

(12)

Here $\bar{D}, (D, D^*)$ represent Livi-Civita connection of $(\bar{M}, g)$ and linear connections on $(TM, S(TM))$ respectively. (B, C) and $(A_X, A^*{}_X)$ represent local fundamental forms and shape operators on $TM$ and $\Gamma(S(TM)$ respectively. Also $\tau$ and $P$ represent 1-form and projection morphisms of $\Gamma(TM)$ on $S(TM)$ respectively. By using the fact that $B(F,J) = g(\overline{D_F} J, \xi)$, we know that the local second fundamental form $B$ is independent of choice of $S(TM)$ and hence satisfies

$$B(F,Z) = 0, \ \forall F \in \Gamma(TM).$$

*(13)*

Unfortunately $D$ on $TM$ is not a metric connection and hence satisfies

$$(D_F \bar{g})(J,L) = B(F,J)\theta(L) + B(F,L)\theta(J) \quad \vee F,J,L \in \Gamma(TM).$$

*(14)*

Here $\theta$ represents 1- form defined as, $\theta(F) = g(F,X)$, for all $F \in \Gamma(TM)$.*TheLiederivativeofgwithrespecttotheitsLevi-Civitaconnection*$\bar{\nabla}$*isdefinedby*

$$(L_F g)(J,L) = g(D_F J, L) + g(D_F L, J) \ \vee F,J,L \in \Gamma(TM).$$

*(15)*

For any lightlike hypersurface $(M, \bar{g}, S(TM))$of $(\bar{M}, g)$, we have from (13) and (14) that

$$(L_F \bar{g})(J,L) = B(F,J)\theta(L) + B(F,L)\theta(J) + \bar{g}(D_F J, L) + \bar{g}(D_F L, J) \ \vee F,J,L \in \Gamma(TM).$$

*(16)*

Or equivalently above equation may be written as

$$(L_F \bar{g})(J,L) = (D_F \bar{g})(J,L) + \bar{g}(D_F J, L) + \bar{g}(D_F L, J) \quad \vee F,J,L \in \Gamma(TM).$$

*(17)*

However, $D^*$ on $S(TM)$ is metric connection, and the above shape operators are related to their local second fundamental forms as

$$B(F,J) = \bar{g}(A^*{}_Z F, J), \ \bar{g}(A^*{}_Z F, X) = 0$$

*(18)*

$$C(F,PJ) = \bar{g}(A_X F, PJ), \ \bar{g}(A_X F, X) = 0.$$

*(19)*

From (16), $A^*{}_Z F$ =0. With respect to connections $\bar{D}$ and $D$, the Riemannian curvature tensors of $(\bar{M}, g)$ and $(M, \bar{g}, S(TM))$ are represented by $\bar{R}$ and $R$ respectively as given by

*(20)* $$g(\bar{R}(F,J)\xi, PO) = \bar{g}(R(F,J)L, PO) + B(F,L)C(J,PO) - B(J,L)C(F,PO)$$

*(21)* $$g(\bar{R}(F,J)L, \xi) = \bar{g}(R(F,J)\xi, PO) = (D\_F B)(J,L) - (D\_J B)(F,L) + B(J,L)\ \tau(F) - B(F,L)\tau(J)$$

*(22)* $$g(\bar{R}(F,J)L, X) = \bar{g}(R(F,J)\xi, X) = \bar{g}(D_F(A_X J) - D_J(A_X F) - \bar{g}(A_X(F,J), L) + \bar{g}(A_X F, L)\tau(J) - \bar{g}(A_X J, L)\tau(F) + \bar{g}(A^*{}_\xi F, A_X J) - \bar{g}(A^*{}_\xi J, A_X F) - 2d\tau(F,J)\theta(L))$$

*(23)* $$(D_F g)(J,L) = B(F,J)\theta(L) + B(F,L)\theta(J) \quad \vee F,J,L \in \Gamma(TM).$$

**Definition:** Let(M, ḡ)bealightlikehypersurfaceofasemi-Riemannianmanifold(M̄, g).Apointp in M is said to be umbilic, if $B_P(F_P, J_P) = k\bar{g}_p(F_P, J_P)$, for $F_p, J_p \in M$, where B is the local secondfundamental form.

One says that $M$ is totally umbilic if any point of $M$ is umbilic.It is easy to see that Mis totally umbilic if and only if, locally, on each there exists a smooth function φsuch that$B(F,J) = \varphi\bar{g}(F,J)$, for any $X$ and $Y$tangent to $M$. In caseφ = 0, one says Mis totallygeodesic.

**Definition:** Let$(M, \bar{g})$bealightlikehypersurfaceofa semi-Riemannianmanifold$(\bar{M}$, g)$. S(TM)$ istotally umbilic, ifonanycoordinateneighborhood Uthereexistsasmoothfunctionψsuchthat$C(F,PJ) = \psi\bar{g}(F,PJ)$, for any $F$ and $J$ tangent to M. In case ψ = 0 (resp. ψ ≠ 0) on U one says $S(TM)$ is totallygeodesic(resp.,properlytotallyumbilic).WecallMatotally screenumbiliclightlikehypersurfaceif,itsscreendistribution is totally umbilic.

**Definition:**A lightlike hypersurface *(M,* $\bar{g}$*)* of a semi-Riemannian manifold is called screen locally conformal, iftheshapeoperators$A_X$and$A_\xi^*$of$M$and$S(TM)$,respectively,arerelatedby$A_X = \phi A_\xi^*$.Where $\phi$is a non-vanishing smooth function on a neighborhood$U$of$M$. We say that$M$is screen homothetic if $\phi$is non-zero constant.

**Definition:**A lightlike hypersurface $(M, \bar{g}, S(TM))$ of a Lorentzian manifold $(\bar{M}, g)$is said to be of genus zero [10] with screen $S(TM)$ if

- $M$admits a canonical or unique screen distribution $S(TM)$ that induces a canonical or unique lightlike transversal vector bundle $N$ and $M$ admits an induced symmetric Ricci tensor.
- Let the Ricci tensor $R^{(0,2)}$ be symmetric on lightlike hypersurface$(M, \bar{g}, S(TM))$.

The manifold $M$ is called as an Einstein lightlike hypersurface [14] if, for any $F,J \in \Gamma(TM)$, the following relation satisfies:

(24) $$R^{(0,2)}(F,J) = \gamma\,\bar{g}(F,J)$$

Where$\gamma$isconstant .

## 3. New Type of Vector Fields

For any lightlike hypersurface $(M, \bar{g}, S(TM))$, some significant differential operators such as the gradient, divergence, Laplacian operators could be defined by the help of a rigging vector field and its associated metric. Therefore, we shall initially recall some basic facts related to rigging vector fields and their some basic properties before studying new type of vector fields on lightlike hypersurfaces.

Let $(M, \bar{g}, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $(\bar{M}, g)$ and $\xi$ be a vector field defined in some open set containing M. suppose that $\xi_p \notin Tp\, M$ for any $p \in M$. If there exists a 1-form $\eta$ satisfying $\eta(F) = g(F, \xi)$ for any $F \in \Gamma(TM)$, then $\xi$ is called a rigging vector field for M.

Now, let $X \in tr(TM)$ be a rigging vector field for $M$ and $\eta$be a 1-form defined by

(25) $$\eta(F) = g(F, X)$$

for any $F \in \Gamma(TM)$. In this case, one can define a (0, 2) type tensor $g$as follows:

(26) $$g(F, J) = \bar{g}(F, J) + \eta(F)\eta(J)$$

for any $X, Y \in \Gamma(TM)$. We note that the associated metric $\bar{g}$is non-degenerate. From (), () and () we have

(27) $$g(\xi, \xi) = 1, \; g(\xi, F) = \eta(F)$$

and

(28) $$\bar{g}(F, J) = g(F, J), \; \eta(F) = 0$$

$\forall F, J \in \Gamma(S(TM))$.

Now, let $v$be a new type of vector field on $\Gamma(T\bar{M})$. Then, we can write $v$as the tangential and transversal components by

(29) $$v = V^T + V^X$$

Where$V^T \in \Gamma(TM)$and $V^X \in tr(TM)$. From () and (), we have

$$g(V^T + \xi) = \bar{g}(V^T, \xi) + \eta(V^T)\eta(\xi))$$

(30) $$\bar{g}(V^T + \xi) = \bar{g}(V^T + X)$$

For any $F \in \Gamma(TM)$, we write

(31) $F\bar{g}(V^T + \xi) = F\bar{g}(V^T + X)$

(32) $= g(D_F v + X) + g(v, D_F X)$

(33) $B(F)\eta(v) - \eta(F) + \tau(F)\eta(v) - g(v, A_X F)$

Now let us assume $v$ lies in tangent bundle i,e. $v = V^T$. So for any $v^S \in \Gamma(S(TM))$ and $\eta(v) = a$, we can write

(34) $$v = v^S + a\xi$$

## 4. Results and Discussion

***Lemma-1.*** Let $v$ be the new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of a semi-Riemannian manifold, then

(35) $$\tau(v) = 1 + \frac{1}{a}[g(v^S, A_N v)]$$

***Proof.***From equations (30), (33) and (34), we acquire

$$(v^S + a\xi)g(v^S + a\xi, \xi) = g(v^S + a\xi, v)g(v^S + a\xi, X) - g(v^S + a\xi, X) - g(v^S + a\xi, A_X(v^S + a\xi)) + \tau(v^S + a\xi)g(v^S + a\xi, X)$$

By straightforward computation we acquire

$$a\tau(v) = a + g(v^S, A_X v) + ag(v^S, A_X \xi)$$

From the above equation we obtain (35).

***Lemma 2.*** Let $v$ be the new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of a semi Riemannian manifold, then for any $F \in \Gamma(TM)$

(36) $$D_F v = B(F)v - F$$

$$B(F, v) = 0$$

***Proof.***From equations (5) and (9) we obtain

(37) $$B(F)V - F = D_F V + B(F, v)F$$

Comparing tangential and transversal parts of the above equation (37) we obtain equations (36) and (37).

***Lemma 3.*** Let $v$ be the new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface of a semi Riemannian manifold, then for any $F \in \Gamma(TM)$, $\tau(v) = 1$.

***Proof.***From equations (9), (32) and lemma-1 we obtain

(38) $$\tau(v) = \frac{1}{a}C(v, V^S) + 1$$

As assumed that is screen conformal null hypersurface we obtain from equation (38) as

(39) $$\tau(v) = \frac{1}{a}\frac{1}{\varphi}B(v, V^S) + 1$$

Using equation (36) in equation (39) we obtain our required result.

***Lemma 4.*** Let $v$ be the new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface of a semi Riemannian manifold, then

(40) $$D_\xi^* v = B(\xi)v \quad and \quad C(\xi, v) = -1.$$

***Proof.*** As second fundamental form $B$ vanishes on $Rad(TM)$. Then using the fact $D_F J = D_F^* J + C(F,J)\xi$, we obtain

$$\bar{D}_\xi v = D_\xi v$$

$$D_\xi v = D_\xi^* v + C(\xi, v)\xi$$

using equations (5) in the above equation, we acquire

(41) $$B(\xi)v - \xi = D_\xi^* v - C(\xi, v)$$

Comparing like terms of the above equation (41), we obtain (40).

***Theorem 1.*** Let $v$ be the new type of vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface, then either $a\tau(\xi) = 2$.

**Proof**. From lemma 3 and the Guass Weingarten formulas we obtain

(42) $$D_v \xi = -A_\xi^* v + \xi$$

(43) $$D_\xi X = -A_X v - \xi$$

Since $v$ is a new type of vector field. Therefore using lemma 4 we obtain

(44) $$D_\xi v = B(\xi)v - \xi$$

From equations (42) and (44) we obtain

(45) $$D_\xi v - D_v \xi = B(\xi)v - \xi + A_\xi^* v - \xi$$

(46) $$[\xi, v] = B(\xi)v + A_\xi^* v - 2\xi$$

If $v = v^S + a\xi$, then above equation becomes

(47) $$[\xi, v^S + a\xi] = B(\xi)(v^S + a\xi) + A_\xi^*(v^S + a\xi) - 2\xi$$

(48) $$[\xi, v^S] = B(\xi)(v^S) + a\xi B(\xi) + A_\xi^* v^S + aA_\xi^* \xi - 2\xi$$

(49) $$D_\xi v^S - D_{v^S}\xi = B(\xi)v^S + B(\xi)a\xi + A_\xi^* v^S + aA_\xi^* \xi - 2\xi$$

Using Guass and Weingarten formulas in above equation we obtain

(50) $$D_\xi^* v^S + C(\xi, v^S)\xi - A_\xi^* v^S + \tau(v^S)\xi = B(\xi)v^S + B(\xi)a\xi + A_\xi^* v^S + aA_\xi^* \xi - 2\xi$$

Since $A_\xi^* \in S(TM)$, therefore from equation (50), we obtain

(51) $$\tau(v^S)\xi + C(\xi, v^S)\xi = -2\xi$$

Using lemma 4

(52) $$\tau(v^S) = -1$$

Using $\tau(v) = 1$ in the above equation we obtain

(53) $$\tau(v) = \tau(v^S + a\xi) = 1$$

Therefore our result follows from above equation.

Now we shall investigate new type of vector fields on the Levi-Civita connection $\bar{D}$ with respect to the associated metric $g$.

Let $\bar{D}$ be the Riemannian connection of $\bar{M}$ with respect to the associated metric $g$ given in the Equation (26) and $D^1$ be the induced Riemann connection from $\bar{D}$ onto *TM*.

***Theorem 2.*** Let $v$ be a be a new type of vector field with respect to $\bar{D}$. Then $v$ is also be a new type of with respect to $D^1$.

**Proof**. From equation (26), for any $X \in \Gamma(TM)$, we have

$$Fg(v, v) = F\bar{g}(v, v) + F(g(v, N)^2)$$

(54) $$2g(D_F v, v) + 2\eta(v)[\eta(V)B(F) - \eta(F) - g(v, A_X F) + \tau(F)\eta(v)]$$

Using the fact

$$Xg(v, v) = 2\bar{g}(D_F^1 v, v)$$

(55) $$Fg(v, v) = 2\bar{g}(D_F^1 v, v) + 2\eta(D_F^1 v)\eta(v)$$

From equations (54) and (55), we acquire

(56) $$D_X^1 v + \eta(D_X^1 v)N = B(X)v - X + [B(X)\eta(v) - \eta(X) - g(v, A_N X) + \tau(X)\eta(v)]N$$

Comparing the corresponding parts on both sides we obtain

(57) $$D_F^1 v = B(F)v - F$$

and

(58) $$\eta(D_F^1 v) = B(F)\eta(v) - \eta(F) - g(v, A_X F) + \tau(F)\eta(v)$$

Equation (57), shows that $v$ is also a new type of vector field with respect to $D^1$. From equation (58) we may state

***Corollary1.*** Let $v$ be a be a new type of vector field with respect to $D^1$. Then $v$ is also be a new type of vector field with respect to $D$ if and only if the following condition

$$\eta(D_F^1 v) = B(F)\eta(v) - \eta(F) - g(v, A_X F) + \tau(F)\eta(v)$$

Is satisfied for all $X \in \Gamma(TM)$.

***Corollary 2.*** Let $v$ be the new type of vector field with respect to $D^1$ lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be screen conformal null hypersurface with respect to $D$. Then $v$ is also be a new type of vector field with respect to $D$ if and only if the following condition

$$\eta(D_V^1 v) = \eta(v) - g(v, A_X F).$$

**5. Ricci Solitons on Null Hypersurfaces**

Let $(M,\bar{g},S(TM))$ be the null hypersurface of $(\bar{M},g)$ and $v$ be the new type of vector field on tangent bundle of $(\bar{M},g)$. Then $v$ can be written as

(59) $$v = V^T + fX.$$

Where $V^T$ and $fX$ are tangential and transversal components of $v$. Also $V^T \in \Gamma(TM)$and $f = g(v,\xi)$.

***Theorem 3.*** Let $v$ be the new type of vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$, then

(60) $$D_F v^T = B(F)v - F + fA_X F$$

(61) $$B(F, v^T) = f\tau(F) - Ff$$

***Proof***. From equation (59)

(62) $$\tau(v) = \tau(V^T + fN)$$

As assumed $v$ is a new type of vector field

(63) $$\bar{D}_F v = \bar{D}_F v^T + \bar{D}_F(fX)$$

(64) $$B(F)v - X = \bar{D}_F v^T + \bar{D}_F(fN)$$

Using Guass and Weingarten formulas in the forgoing equation we obtain

(65) $$\bar{D}_F v = D_F v^T + B(F, v^T)N + F(f)N - fA_X F - f\tau(F)N$$

From equations (63) and (65), we obtain equations (60) and (61).

***Theorem 4.*** Let $v$ be the new type of vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be a totally geodesic null hypersurface of $(\bar{M},g)$. Then either the function $f$ is constant or $(M,\bar{g},S(TM))$ is integrable.

***Proof.***From equation (61), if null (lightlike) hypersurface $(M,\bar{g},S(TM))$ is totally geodesic then we acquire

(66) $$f\tau(F) = Ff$$

Similarly

(67) $$f\tau(J) = Jf$$

Using equations (67) and (68), we obtain

(68) $$[F,J](f) = f\{J(\tau(F)) - F(\tau(J))\}$$

For any $X, Y \in \Gamma(TM)$.

Using corollary 1 and proposition 2 of [18] in equation (68), we obtain, $[F,J](f) = 0$. This signifies that either the function $f$ is constant or $(M,\bar{g},S(TM))$ is integrable.

**Lemma 5.** Let $v$ be the new type of vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$, thenfor any $F, J \in \Gamma(TM)$ we have

(69) $$(L_{v^T})\bar{g}(F,J) = B(v^T,F)\theta(J) + B(v^T,J)\theta(F) + 2B(F)B(J) - 2\bar{g}(F,J) + f\bar{g}(A_N F,J) + f\bar{g}(A_N J,F)$$

or equivalently

(70) $$(L_{v^T})\bar{g}(F,J) = (D_{v^T}\bar{g})(F,J) + 2[B(F)B(J) - \bar{g}(F,J)] + f[\bar{g}(A_N F,J) + \bar{g}(A_X J,F)]$$

***Proof.*** From equations (16) and (17), we obtain

(71) $$(L_{v^T}\bar{g})(F,J) = B(v^T,F)\theta(J) + B(v^T,J)\theta(F) - \bar{g}(D_F v^T,J) - -\bar{g}(D_J v^T,F)$$

or equivalently

(72) $$(L_{v^T}\bar{g})(F,J) = (D_{v^T}\bar{g})(F,J) + \bar{g}(D_F v^T,J) + \bar{g}(D_J v^T,F)$$

By making use of equations (59) and (60) in equations (71) and (72), we obtain our results.

***Lemma 6***. Let $v$ be the new type of vector field lying on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$, thenfor any $X, Y \in \Gamma(TM)$ we have

(73) $$(L_{v^T}\bar{g})(F,J) = (D_{v^T}\bar{g})(F,J) + 2[B(F)B(J) - \bar{g}(F,J).$$

***Proposition 1***. Let $v^T$ be a potential vector field and $(M,\bar{g},S(TM))$ be null hypersurface of $(\bar{M},g)$. Thenfor any $X, Y \in \Gamma(TM)$, we have

(74) $$2R^{(0,2)}(F,J) = 2\lambda\bar{g}(F,J) - f[\bar{g}(A_X F,J) + \bar{g}(A_N J,F)] - 2[B(F)B(J) - \bar{g}(F,J) - (D_{v^T}\bar{g})(F,J).$$

Or equivalently

(75) $$R^{(0,2)}(F,J) = (\lambda - 1)\bar{g}(F,J) - \frac{f}{2}[\bar{g}(A_X F,J) + \bar{g}(A_X Y,F)] - B(F)B(J) - \frac{1}{2}(D_{v^T}\bar{g})(F,J).$$

***Proof.***We know that if $(M,\bar{g},S(TM))$ is Ricci soliton lightlike hypersurface then it satisfies

(76) $$(L_{v^T}\bar{g})(F,J)+2R^{(0,2)}(F,J)=2\lambda\bar{g}(F,J).$$

Using lemma 5 in the above equation we obtain our required results.

***Proposition 2***. Let $v$ be a potential vector field and $(M,\bar{g},S(TM))$ be null (lightlike) hypersurface of semi-Euclidean space. Thenfor any $F,\in \Gamma S((TM))$, we have

(77) $$nB(F,F)traceA_X-B(A_XF,F)=\lambda-1-f\bar{g}(A_XF,F)-(B(F))^2.$$

Where $F$ is a unit vector field.

***Proof.*** Assume that ($a_1, a_2, a_3\ldots\ldots\ldots a_n$) is an orthonormal basis. Therefore from equation (83) of [18], we obtain

(78) $$R^{(0,2)}(a_i,a_i)=nB(a_i,a_i)traceA_N-B(A_Na_i,a_i)$$

Using equation (75) in equation (78) we obtain

(79) $$nB(a_i,a_i)traceA_N-B(A_Na_i,a_i)=(\lambda-1)\bar{g}(a_i,a_i)-f\bar{g}(A_Na_i,a_i)-(B(a_i))^2-\frac{1}{2}(D_{v^T}\bar{g})(a_i,a_i).$$

Replacing $a_i$with $F$, we obtain

(80) $$nB(F,F)traceA_N-B(A_NF,F)=\lambda-1-f\bar{g}(A_XF,F)-(B(F))^2.$$

***Corollary 3***. Let $v$ be a new type of vector field and $(M,\bar{g},S(TM))$ be Ricci soliton null (lightlike) hypersurface of semi-Euclidean space. Thenfor any $F,J\in \Gamma S((TM))$,if $(M,\bar{g},S(TM))$ is screen conformal and totally umbilical then we have

(81) $$n^2[B(F,F)]^2\varphi-[B(F,F)]^2=\lambda-1-2fB(F,F)-(B(F))^2$$

where$F$ is a unit vector field.

***Proposition 3.***Let $v$ be a new type of vector field on $\Gamma(TM)$ and $(M,\bar{g},S(TM))$ be Ricci soliton null (lightlike) hypersurface of a Lorentzian manifold. Thenfor any $F,J\in \Gamma S((TM))$,we have

(82) $$R^{(0,2)}(F,J)=(\lambda-1)\bar{g}(F,J)-\frac{1}{2}[B(v,F)\theta(J)+B(v,J)\theta(F)]-B(F)B(J).$$

***Proof.***Since $(M,\bar{g},S(TM))$ is a Ricci soliton and $v$ is a new type of vector field lying on tangent bundle. Therefore equation (76) can be written as

(83) $$(L_v\bar{g})(F,J)+2R^{(0,2)}(F,J)=2\lambda\bar{g}(F,J).$$

For any $X,Y\in\Gamma(TM)$ and from lemma 6, we have

(84) $$(L_v\bar{g})(F,J)=(D_v\bar{g})(F,J)+2[B(F)B(J)-\bar{g}(F,J).$$

Using equations (83) and (84), we obtain

(85) $$R^{(0,2)}(F,J)=(\lambda-1)\bar{g}(F,J)-\frac{1}{2}D_v\bar{g}(F,J)-B(F)B(J).$$

Therefore from equations (14) and (84) the proof is straightforward.

## References


[1] H. AlSodais, H. Alodan, S. Deshmukh, *Hypersurfaces of Euclidean space as gradient Ricci solitons.* Int. J. Pure Appl. Math.2014, **1**, 437–444.

[2] C. Atindogbe, K. L. Duggal, *Conformal screen on lightlike hypersurfaces*. Int. J. Pure Appl. Math. 2004, **11**, 421–442.

[3] C. Atindogbe, J. P. Ezin, J. Tossa, *Lightlike Einstein hypersurfaces in Lorentzian manifolds with constant curvature*. Kodai Math. J. 2006, **29**, 58–71

[4] C. L. Bejan, M, Crasmareanu. *Ricci solitons in manifolds with quasi-constant curvature*. Publ. Math. Debr. 2011, **78**, 235–243

[5] A. M. Blaga, S. Y. Perkta, B. E. Acet, F. E Erdogan, *h-Ricci solitons in almost paracontact metric manifolds*. Glasnik Matematicki 2018, **53**, 205–220.

[6] F. Brickell, K. Yano, *Concurrent vector fields and Minkowski structures*. In Kodai Mathematical Seminar Reports; Department of Mathematics, Tokyo Institute of Technology, Tokyo, Japan, 1974, **26**, 22–28.

[7] G. Catino, L. Mazzieri,*Gradient Einstein solitons*, Nonlinear Analysis, 2016, **132**, 66-94.

[8] B.Y. Chen, *A simple characterization of generalized Robertson-Walker spacetime. Gen. Relat. Gravitation***46**(12), 5 (2014)

[9] B. Y. Chen, S. Deshmukh, *Ricci solitons and concurrent vector fields*. Balk. J. Geom. Appl. 2015, **20**, 14–25.

[10] U. C. De, A. Sarkar, K. DE, Ricci-Yambe solitons and 3-dimensional manifolds. Turk. j. Math. 2022, **46,** 1078-1088.

[11] K. L. Duggal, *On scalar curvature in lightlike geometry*. J. Geom. Phys. 2007, **57**, 473–481.

[12] K. L. Duggal, A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications;* Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1996.

[13] K. L. Duggal, D. H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific Publishing: Hackensack, NJ, USA, 2007.

[14] K. L. Duggal, D. H. Jin, *Totally umbilical lightlike submanifolds*. Kodai Math. J. 2003, **26**, 49–68.

[15] K. L. Duggal, B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Springer Science, Business Media: Berlin, Germany, 2011.

[16] S. Guler, M. Crasmareanu,*Ricci-Yamabe maps for Riemannian flow and their volume variation and volume entropy*, Turkish Journal of Mathematics, 2019, **43,** 2631-2641.

[17] R. S. Hamilton, *The Ricci flow on surfaces*. Mathematics and general relativity, 1998, **71**, 237-262.

[18] R. S. Hamilton, Lecture *on geometric flows*. 1989 (unpublished).

[19] E. Kilic, M. Gulbahar, and E. Kavuk, Concurrent Vector Fields on Lightlike Hypersurfaces. Mathematics, 2021, **9,** 59, 1-11.

[20] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer Academic: Dordrecht, The Netherlands, 1996.

[21] F. Ngakeu, H. F. Tetsing, *a-Associated metrics on rigged null hypersurfaces*. Turk J. Math. 2019, **43**, 1161–1181.

[22] B. Olea, *Null Hypersurfaces on Lorentzian Manifolds and Rigging Techniques*, Springer Proceedings in Mathematics & Statistics; Springer: Cham, Switzerland, 2016, **211**, 237–251.

[23] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*. arXiv2002, arXiv:math/0211159.

[24] H. Takeno, *Concircular scalar field in spherically symmetric space-times* I. Tensor **20**(2), 167–176 (1967)

[25] H. F. Tetsing, F. Ngakeu, B. Olea, *Rigging technique for 1-lightlike submanifolds and preferred rigged connections*. Med. J. Math. 2019, **16**, 139.

[26] K. Yano, *Concircular geometry. I. Concircular transformations*. Proc. Imp. Acad. Tokyo **16**, 195–200 (1940)

[27] K. Yano,*Integral Formulas in Riemannian geometry*. Marcel Dekker, 1970.

[28] K. Yano, *Sur le parallelisme et la concourance dans l'espace de Riemann*. Proc. Imp. Acad. 1943, **19**, 189–197.