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Coefficients Estimates of Bi-Univalent Functions Defined by Quasi-Subordination

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Abstract: In the present paper ,realization new classes $\mathcal{W}_{\beth}^{q}(\alpha,\beta,h)$ and $\mathcal{F}_{\beth}^{q}(\alpha',\beta',h)$ of bi - univalent functions defined in the open unit disk U and its inverse $g=f^{-1}$ satisfying the conditions that with quasi - subordination is defined on the first two Taylor - Maclaunin series coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses are determined . Several special consequences of the results are also indicated.

Keyword: bi-univalent, quasi-subordination,univalent function, starlike function,convex function,

1-Introduction:

Let H be class of analytic functions f defined in an open unit disk $U=\{z: z\in\mathbb{C}\ and\ |z|<1\}$ and normalized by the conditions $f(0)=0\,, f'(0)=1$ in U. An analytic function $f\in H$ has Taylor series expansion of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k\ ,(z\in U). \quad (1.1)$$

Further , let $\mathcal{A}$ symbol the class of all functions in H consisting of form (1.1) which are univalent functions in U. For two analytic functions f and Φ, the function f is said to be subordinate to Φ in U and written as $f(z)\prec\Phi(z)$, if there exists a Shwarz function w be analytic such that $f(z)=\Phi(w(z))$ with $w(0)=0$ and $|w(z)|\leq 1,(z\in U)$.

The Koebe - One - Quarter Theorem [11] ensures that the image of U under every univalent function $f\in\mathcal{A}$ contains a disk of radius $\frac{1}{4}$.

Thus every univalent function f has an inverse f^{-1} is satisfying ([3] and [14]):

$$f^{-1}(f(z)) = z\,,(z\in U),$$

and

$$f(f^{-1}(w)) = w,(|w|<r_0(f)\,, r_0(f)\geq\tfrac{1}{4}),$$

where

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^2-5a_2a_3+a_4)w^4+\dots,(w\in U). \quad (1.2)$$

A function $f\in H$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let $\beth$ denote the class of bi-univalent functions defined in U.

In the year 1970, [25] proposed the notion of $quasi-subordination$ for the first time. The function f is said to be $quasi-subordinate$ to in U for two analytic functions f and Φ and is expressed as

$$f(z)\prec_q \Phi(z),(z\in U),$$

if there exists analytic functions in U , $\vartheta(z)$ and $w(z)$, $w(0)=0$ such that $|\vartheta(z)|<1, |w(z)|<1$ and $f(z)=\vartheta(z)\,\Phi(w(z))$, for all $z\in U$. If $\vartheta(z)=1$, $then$ $f(z)=\Phi(w(z))$, so that $f(z)\prec\Phi(z)$ in U. Also notice that if $w(z)=z$, then $f(z)=\vartheta(z)\,\Phi(z)$and it is said that f is $majorized$ by Φ and written $f(z)\ll\Phi(z)$ in U (see [11]) . Hence it is $obvious$ that $quasi-subordination$ is a $generalization$ of $subordination$ as $well$ as $majorization$ ([4,26 ,25, 27]) .

In the sequel , it assumed that Φ is analytic in U satisfying $\Phi(0)=1,\Phi'(0)>0$ such that a function has Taylor a $series$ $expansion$ of the form:

$$\Phi(z)=1+\sum_{j=2}^{\infty}c_jz^j\ (c_1>0) \quad (1.3)$$

and

$$\vartheta(z)=K_0+K_1z+K_2z^2+\cdots, \quad (1.4)$$

which analytic and bounded in U. However , there $are\ only\ a\ few\ works\ deter$mining the general coefficient bounds $|a_2|$and $|a_3|$

([1,2,5,6,7,8,9,12,13,15,16,28,29,30,31,32]) for the analytic bi- univalent functions in the literature . ([8,9,10,11]) Ma and $Minda$ [18] defined a class of $starlike$ and convex functions for quantities $\frac{zf'(z)}{f(z)}$ and $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function and using the method of subordination , and studied a class $S^*(\Phi)$ which is defined by

$$S^*(\Phi)=\left\{f\in H: \frac{zf'(z)}{f(z)}\prec\Phi(z), z\in U\right\},$$

and

$$G^*(\Phi)=\left\{f\in H: 1+\frac{zf''(z)}{f'(z)}\prec\Phi(z), z\in U\right\}.$$

The functions in the classes $S^*(\Phi)$ and $G^*(\Phi)$ are known as $starlike$ of $Ma-Minda$ type and convex of Ma - Minda type, respectively. $S^*{}_{\beth}(\Phi)$ and $G^*{}_{\beth}(\Phi)$ designate bi - starlike and bi - convex functions f is bi - $starlike$ and bi - convex of $Ma-Minda$ type, respectively [18].

Lewin [17] explored the class ב of bi - univalent functions in 1967 and determined the constraint for the second coefficient a_2. *Brannan* and *Taha* [9] studied subclasses of bi - univalent functions that are analogous to the well-known subclasses of univalent functions, which include *starlike*, highly *starlike*, and convex functions. They developed the bi-starlike function and bi-convex function classes, and derived non-sharp estimates for the first two *Taylor - Maclaurin* coefficients $|a_2|$ and $|a_3|$. Ali et al. [3], D*eniz* [10], Peng et al. [23], Ramchandran et al.[24], Murugusundaramoorthy et al.[20] and others have recently created and analyzed Ma- Minda type subclasses of the bi - univalent function class ב. Several writers, notably ([14], [19]), have produced further generalizations of the *Ma - Minda* type subclasses of class ב by using *quasi – subordination*. Motivated by our work on *quasi-subordination* in [22], we develop and investigate several new subclasses of class ב.

Let $h(z)$ be analytic in $\mathbb{U}$ with $h(z) = 1$ and

$$\mathcal{g}(z) = A_0 + A_1 z + A_2 z^2 + \cdots ; \quad (|\mathcal{g}(z)| < 1, z \in \mathbb{U}) \qquad (1.5)$$

$$h(z) = 1 + B_0 + B_1 z + B_2 z^2 + \cdots ; \quad (B_1 > 0\ ). \qquad (1.6)$$

Oshah and Darus [21] defined the following generalized derivative operator:

$$\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell\big(1 + (\lambda_1 + \lambda_2)(k-1)\big) + d}{\ell\big(1 + \lambda_2(k-1)\big) + d}\right]^m z^k$$

where $f(z) \in \mathcal{A}, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0\ and\ \ell + d > 0$.

Definition 1.1. If $f \in$ ב, then $f \in \mathcal{W}^q_{ב}(\alpha,\beta,h)$ $\alpha \geq 0, 0 \leq \beta \leq 1)$ if the following quasi-subordination hold:

$$\left[\left(\frac{z\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)\big)''}{\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)\big)'} - 2\alpha\beta\right) + (\alpha\beta - 1)^2 \frac{z\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)\big)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)} - \alpha^2\beta^2 \frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right] - 1 \prec_q (h(z) - 1),$$

$$\left[\left(\frac{w\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} g(w)\big)''}{\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} g(w)\big)'} - 2\alpha\beta\right) + (\alpha\beta - 1)^2 \frac{w\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} g(w)\big)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} g(w)} - \alpha^2\beta^2 \frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} g(w)}{w}\right] - 1 \prec_q (h(w) - 1).$$

Definition 1.2. If $f \in$ ב, then $f \in \mathcal{F}^q_{ב}(\alpha',\beta',h)$ if the following quasi − subordination hold

$(\alpha' \geq 1, \beta' = 0,1,2,3 \ldots)$

$$\left((\alpha'\beta' + 1)\frac{z\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)\big)'}{z} + \alpha'\beta' z\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)\big)'' + \frac{-\alpha'\beta'\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right) - 1 \prec_q (h(z) - 1),$$

$$\left((\alpha'\beta' + 1)\frac{w\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(w)\big)'}{w} + \alpha'\beta' w\big(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(w)\big)'' + \frac{-\alpha'\beta'\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d} f(w)}{w}\right) - 1 \prec_q (h(w) - 1).$$

Lemma 1. 3 [3]. If $p \in P$, then $|p_i| \leq 2$ for each i, where P is the family of all functions p , analytic in U, for which $Re(p(z)) > 0$, where $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots, for\ z \in \mathbb{U}$.

2. Main Results:

Theorem 2.1. If $f \in \mathcal{W}^q_{ב}(\alpha,\beta,h)$ $(\alpha \geq 0, 0 \leq \beta \leq 1)$, then

$$|a_2| \leq \min\left\{\frac{|A_0|B_1}{8|3-2\alpha\beta|\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{2|\alpha\beta|\left|\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right|}}\right\} \qquad (2.1)$$

and

$$|a_3| \leq min\left\{\frac{[|A_1| + |A_1|]B_1}{|\alpha^2\beta^2 + 2\alpha\beta + 8|\left|\left[\left[\frac{\ell\big(1 + (\lambda_1 + \lambda_2)(k-1)\big) + d}{\ell\big(1 + \lambda_2(k-1)\big) + d}\right]^m\right]^2\right|} + \frac{|A_0|^2 B_1^2 (c_1^2 + d_1^2)}{2|3 - 2\alpha\beta|^2 \left[\left[\frac{\ell\big(1 + (\lambda_1 + \lambda_2)(k-1)\big) + d}{\ell\big(1 + \lambda_2(k-1)\big) + d}\right]^m\right]^2}, \frac{[|A_1| + |A_1|]B_1}{|\alpha^2\beta^2 + 2\alpha\beta + 8|\left|\left[\left[\frac{\ell\big(1 + (\lambda_1 + \lambda_2)(k-1)\big) + d}{\ell\big(1 + \lambda_2(k-1)\big) + d}\right]^m\right]^2\right|} + \frac{|A_0|(B_1 + |B_2 - B_1|)}{|2\alpha\beta|\left[\frac{\ell\big(1 + (\lambda_1 + \lambda_2)(k-1)\big) + d}{\ell\big(1 + \lambda_2(k-1)\big) + d}\right]^m}\right\} \qquad (2.2)$$

Proof :

Let $f \in \mathcal{W}^q(\alpha,\beta,h)$, there exist the Schwarz functions $\mathcal{R}(z), \Im(z)\ with$

$$\mathcal{R}(z) = c_1 z + \sum_{j=2}^{\infty} c_j z^j, \qquad (z \in \mathbb{U}\ )$$

$$\Im(z) = d_1 z + \sum_{j=2}^{\infty} d_j z^j, \qquad (z \in \mathbb{U}\ )$$

$\mathcal{R}(0) = \Im(0) = 0\ and\ |\mathcal{R}(z)| < 1, |\Im(w|) < 1$ and an analytic function $\mathcal{g}(z)$ such that

$$\left[\left(\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)''}{\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}-2\alpha\beta\right) + (\alpha\beta-1)^2\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)} - \alpha^2\beta^2\frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right]-1 = g(z)(h(\mathcal{R}(z))-1) \quad (2.3)$$

$$\left[\left(\frac{w\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)\right)''}{\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)\right)'}-2\alpha\beta\right) + (\alpha\beta-1)^2\frac{w\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)\right)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)} - \alpha^2\beta^2\frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)}{w}\right]-1 = g(w)(h(\Im(w))-1). \quad (2.4)$$

Define the functions

$$p(z)=\frac{1+\mathcal{R}(z)}{1-\mathcal{R}(z)}=c_1z+c_2z^2+\cdots \quad (2.5)$$

$$q(w)=\frac{1+\Im(w)}{1-\Im(w)}=d_1w+d_2w^2+\cdots \quad (2.6)$$

or equivalently

$$\mathcal{R}(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1}{2}\right)z^2+\cdots\right] \quad (2.7)$$

$$\Im(z)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left[d_1w+\left(d_2-\frac{d_1}{2}\right)w^2+\cdots\right]. \quad (2.8)$$

It is clear that $p(z), q(w)$ are analytic and have positive real parts in $\mathbb{U}$. In view of (2.3), (2.4), (2.7) and (2.8) clearly

$$\left[\left(\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)''}{\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}-2\alpha\beta\right) + (\alpha\beta-1)^2\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)} - \alpha^2\beta^2\frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right]-1 = g(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right) \quad (2.9)$$

$$\left[\left(\frac{w\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)\right)''}{\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)\right)'}-2\alpha\beta\right) + (\alpha\beta-1)^2\frac{w\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)\right)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)} - \alpha^2\beta^2\frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(w)}{w}\right]-1 = g(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right)-1\right), \quad (2.10)$$

where $f(z)$ and g(w) as given in (1.1) and (1.2) respectively.

$$\left[\left(\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)''}{\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}-2\alpha\beta\right) + (\alpha\beta-1)^2\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)} - \alpha^2\beta^2\frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right]-1 = (3-2\alpha\beta)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^m a_2z + [(\alpha^2\beta^2-4\alpha\beta+8)a_3 + (-\alpha^2\beta^2-2\alpha\beta-5)a_2^2]\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^m z^2+\cdots \quad (2.11)$$

$$\left[\left(\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)''}{\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}-2\alpha\beta\right) + (\alpha\beta-1)^2\frac{z\left(\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)} - \alpha^2\beta^2\frac{\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right]-1 = -(3-2\alpha\beta)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^m a_2z + [(\alpha^2\beta^2-6\alpha\beta+5)a_2^2 + (-\alpha^2\beta^2+4\alpha\beta-8)a_3]\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^m z^2+\cdots. \quad (2.12)$$

Using (2.5) and (2.6) together with (1.5) and (1.6)

$$g(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right) = \frac{1}{2}A_0B_1c_1z + \left[\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2+\frac{c_1^2}{2}\right) + \frac{A_0B_2c_1^2}{4}\right]z^2 \quad (2.13)$$

$$g(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right)-1\right) = \frac{1}{2}A_0B_1d_1z + \left[\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2+\frac{d_1^2}{2}\right) + \frac{A_0B_2d_1^2}{4}\right]z^2. \quad (2.14)$$

From (2.9) we get (2.11) equal (2.13)

$$(3-2\alpha\beta)\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m a_2 = \frac{1}{2}A_0B_1c_1 \quad (2.15)$$

$$[(\alpha^2\beta^2-4\alpha\beta+8)a_3 + (-\alpha^2\beta^2-2\alpha\beta-5)a_2^2]\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m = \frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2+\frac{c_1^2}{2}\right) + \frac{A_0B_2c_1^2}{4}. \quad (2.16)$$

Similarly, (2.10) we get (2.12) equal (2.14)

$$-(3-2\alpha\beta)\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m a_2 = \frac{1}{2}A_0B_1d_1 \quad (2.17)$$

$$[(\alpha^2\beta^2-6\alpha\beta+5)a_2^2 + (-\alpha^2\beta^2+4\alpha\beta-8)a_3]\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m = \frac{1}{2}A_0B_1d_1+\frac{1}{2}A_0B_1\left(d_2+\frac{d_1^2}{2}\right) + \frac{A_0B_1d_1^2}{4}. \quad (2.18)$$

From (2.15) and (2.17), we find

$$c_1=-d_1 \quad (2.19)$$

$$a_2^2 = \frac{A_0^2B_1^2(c_1^2+d_1^2)}{8(3-2\alpha\beta)^2\left[\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right]^2}. \quad (2.20)$$

Adding (2.16), (2.18), we get

$$a_2^2 = \frac{2A_0B_1(c_1+d_1)+A_0(B_2-B_1)(c_1^2+d_1^2)}{-16\alpha\beta\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m}. \quad (2.21)$$

Lemma (1.3) is applied for c_1, c_2, d_1 dy and d_2 follows from (2.20),(2.21), we get

$$|a_2| \le \frac{|A_0|B_1}{8|3-2\alpha\beta|\left|\left[\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right]^2\right|}$$

$$|a_2| \le \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{2|\alpha\beta|\left|\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right|}}$$

$$|a_2| \le min\left\{\frac{|A_0|B_1}{8|3-2\alpha\beta|\left|\left[\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right]^2\right|}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{2|\alpha\beta|\left|\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right|}}\right\} \quad (2.22)$$

That provided $|a_2|$ as showed (2.1).

New further computations (2.16) to (2.18) lead to

$$a_3 = \frac{4A_1B_1c_1(c_1+d_1)+2A_0B_1(c_2-d_2)(c_1^2+d_1^2)}{2(\alpha^2\beta^2+2\alpha\beta+8)\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m} + a_2^2.$$

Upon substituting the value of a from (2.20), (2.21) and Lemma (1.3) is applied for c_1, c_2, d_1 and d_2, we get

$$|a_3| \le \frac{[|A_1|+|A_1|]B_1}{|\alpha^2\beta^2+2\alpha\beta+8|\left|\left[\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right]^2\right|} + \frac{|A_0|^2B_1^2(c_1^2+d_1^2)}{2|3-2\alpha\beta|^2\left[\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right]^2}$$

$$|a_3| \le \frac{[|A_1|+|A_1|]B_1}{|\alpha^2\beta^2+2\alpha\beta+8|\left|\left[\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m\right]^2\right|} + \frac{|A_0|(B_1+|B_2-B_1|)}{|2\alpha\beta|\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(k-1)\big)+d}{\ell\big(1+\lambda_2(k-1)\big)+d}\right]^m}$$

$$|a_3| \le \min\left\{\frac{[|A_1|+|A_1|]B_1}{|\alpha^2\beta^2+2\alpha\beta+8|\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|} + \frac{|A_0|^2B_1^2(c_1^2+d_1^2)}{2|3-2\alpha\beta|^2\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2}, \frac{[|A_1|+|A_1|]B_1}{|\alpha^2\beta^2+2\alpha\beta+8|\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|} + \frac{|A_0|(B_1+|B_2-B_1|)}{|2\alpha\beta|\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m}\right\}. \tag{2.23}$$

That provided $|a_3|$ as showed (2.2). If putting $\alpha = 1,\ \beta = 1$ in Theorem 2.1, we get

Corollary 2.2: Let $f \in \mathcal{W}_{\beth}^{q}(0,\beta,h)$. Then

$$|a_2| \le \min\left\{\frac{|A_0|B_1}{8\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|} \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{2\left|\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right|}}\right\}$$

$$|a_3| \le \min\left\{\frac{[|A_1|+|A_1|]B_1}{11\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|} + \frac{|A_0|^2B_1^2(c_1^2+d_1^2)}{2\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2}, \frac{[|A_1|+|A_1|]B_1}{11\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|} + \frac{|A_0|(B_1+|B_2-B_1|)}{2\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m}\right\}$$

If putting $\mathcal{g}(z) = 1$ in Theorem 2.1, we get

Corollary 2.3. Let $f \in \mathcal{W}_{\beth}^{q}(\alpha,\beta,h)$. Then

$$|a_2| \le \min\left\{\frac{B_1}{8|3-2\alpha\beta|\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|}, \sqrt{\frac{(B_1+|B_2-B_1|)}{2|\alpha\beta|\left|\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right|}}\right\}$$

$$|a_3| \le \min\left\{\frac{B_1}{|\alpha^2\beta^2+2\alpha\beta+8|\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|} + \frac{B_1^2}{2|3-2\alpha\beta|^2\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2}, \frac{B_1}{|\alpha^2\beta^2+2\alpha\beta+8|\left|\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|} + \frac{(B_1+|B_2-B_1|)}{|2\alpha\beta|\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m}\right\}.$$

Theorem 2.4. If $f \in \mathcal{F}_{\beth}^{q}(\alpha',\beta',h)$, $(\alpha' \ge 1, \beta' = 0,1,2,3 \ldots)$, then

$$|a_2| \le \frac{2|A_0|B_1\sqrt{B_1}}{\sqrt{4(16\alpha\beta-6)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m A_0{B_1}^2 - (B_2-B_1)4\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2(3\alpha\beta+2)^2}} \tag{2.24}$$

and

$$|a_3| \le \frac{[|A_1|+|A_0|]B_1}{\left|4(3\alpha\beta+2)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right|} + \frac{|A_0|^2B_1^2}{\left|4(3\alpha\beta+2)^2\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|}. \tag{2.25}$$

Proof Since $\mathcal{F}_{\beth}^{q}(\alpha',\beta',h)$ and $g=f^{-1}$, there exist Schwarz functions $\mathcal{R}(z)$, $\Im(w)$ and an analytic function $g(z)$ such that

$$\left[(\alpha'\beta'+1)\frac{z\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)\right)'}{z}+\alpha'\beta' z\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)\right)''+\frac{-\alpha'\beta'\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)}{z}\right]-1=g(z)\left(h(\mathcal{R}(z))-1\right), \tag{2.26}$$

$$\left[(\alpha'\beta'+1)\frac{w\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)\right)'}{w}+\alpha'\beta' w\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)\right)''+\frac{-\alpha'\beta'\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)}{w}\right]-1=g(w)(h(\Im(w))-1). \tag{2.27}$$

For $p(z)$, $q(w)$as given in (2.5), (2.6) in view of (2.25), (2.26) clearly

$$\left[(\alpha'\beta'+1)\frac{z\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)\right)'}{z}+\alpha'\beta' z\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)\right)''+\frac{-\alpha'\beta'\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)}{z}\right]-1=g(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right), \tag{2.28}$$

$$\left[(\alpha'\beta'+1)\frac{w\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)\right)'}{w}+\alpha'\beta' w\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)\right)''+\frac{-\alpha'\beta'\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)}{w}\right]-1=g(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right)-1\right). \tag{2.29}$$

Since

$$\left[(\alpha'\beta'+1)\frac{z\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)\right)'}{z}+\alpha'\beta' z\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)\right)''+\frac{-\alpha'\beta'\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(z)}{z}\right]-1$$
$$=(3\alpha'\beta'+2)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}a_2z+(6\alpha'\beta'+3)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}a_3z^2+\cdots, \tag{2.30}$$

$$\left[(\alpha'\beta'+1)\frac{w\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)\right)'}{w}+\alpha'\beta' w\left(\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)\right)''+\frac{-\alpha'\beta'\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}f(w)}{w}\right]-1$$
$$=-(3\alpha'\beta'+2)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}a_2w+\left[(16\alpha'\beta'-6)a_2^2-(6\alpha'\beta'+3)a_3\right]\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}w^2+\cdots. \tag{2.31}$$

From (2.28) we get (2.30) equal (2.13)

$$(3\alpha'\beta'+2)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}a_2=\frac{1}{2}A_0B_1c_1 \tag{2.32}$$

$$(6\alpha'\beta'+3)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2+\frac{c_1^2}{2}\right)+\frac{A_0B_2c_1^2}{4}. \tag{2.33}$$

Also from (2.29), we get (2.31) equal (2.14)

$$-(3\alpha'\beta'+2)\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}a_2=\frac{1}{2}A_0B_1d_1 \tag{2.34}$$

$$\left[(16\alpha'\beta'-6)a_2^2-(6\alpha'\beta'+3)a_3\right]\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}=\frac{1}{2}A_0B_1d_1+\frac{1}{2}A_0B_1\left(d_2+\frac{d_1^2}{2}\right)+\frac{A_0B_1d_1^2}{4}. \tag{2.35}$$

From (2.32), (2.34) it follows that

$$c_1=-d_1 \tag{2.36}$$

$$(c_1^2+d_1^2)=\frac{4(3\alpha'\beta'+2)^2\left[\left[\frac{\ell\left(1+(\lambda_1+\lambda_2)(k-1)\right)+d}{\ell\left(1+\lambda_2(k-1)\right)+d}\right]^{m}\right]^2{a_2}^2}{{A_0}^2{B_1}^2}. \tag{2.37}$$

Adding (2.33), (2.35) and using (2.36), (2.37) ,we obtain

$$a_2^2 = \frac{A_0^2 B_1^3 (c_1^2 + d_1^2)}{4(16\alpha'\beta' - 6)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m A_0 {B_1}^2 - (B_2 - B_1)4\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2 (3\alpha'\beta' + 2)^2}. \quad (2.38)$$

Lemma (1.3) is applied for c_1, d_1

$$|a_2| \leq \frac{2|A_0|B_1\sqrt{B_1}}{\sqrt{4(16\alpha'\beta' - 6)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m A_0 {B_1}^2 - (B_2 - B_1)4\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2 (3\alpha'\beta' + 2)^2}} \quad (2.39)$$

That provided $|a_2|$ as showed (2.24).

New further computations (2.33) to (2.35) lead to

$$a_3 = \frac{A_1 B_1 (c_1 - d_1) + 2A_0 B_1 (c_2 - d_2)}{4(3\alpha'\beta' + 2)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m} + \frac{A_0^2 B_1^2 (c_1^2 + d_1^2)}{4(3\alpha'\beta' + 2)^2\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2} \quad (2.40)$$

Lemma (1.3) is applied for c_1, c_2, d_1 and d_2, we get

$$|a_3| \leq \frac{[|A_1| + |A_0|]B_1}{\left|4(3\alpha'\beta' + 2)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right|} + \frac{|A_0|^2 B_1^2}{\left|4(3\alpha'\beta' + 2)^2\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|}. \quad (2.41)$$

That provided $|a_3|$ as showed (2.25).

If putting $\alpha' = 1, \beta' = 1$ in Theorem 2.4, we get

Corollary 2.5. Let $f \in \mathcal{F}_{\beth}^{q}(\alpha', \beta', h)$. Then

$$|a_2| \leq \frac{2|A_0|B_1\sqrt{B_1}}{\sqrt{40\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m A_0 {B_1}^2 - 144(B_2 - B_1)\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2}},$$

and

$$|a_3| \leq \frac{[|A_1| + |A_0|]B_1}{\left|24\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right|} + \frac{|A_0|^2 B_1^2}{\left|144\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|}.$$

If putting $g(z) = 1$ in Theorem 2.4, we get

Corollary 2.6. Let $f \in \mathcal{F}_{\beth}^{q}(\alpha', \beta', h)$. Then

$$|a_2| \leq \frac{2B_1\sqrt{B_1}}{\sqrt{\left|4(16\beta'\alpha' - 6)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m {B_1}^2 - (B_2 - B_1)4\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2 (3\alpha'\beta' + 2)^2\right|}}$$

and

$$|a_3| \leq \frac{B_1}{\left|4(3\alpha'\beta' + 2)\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right|} + \frac{B_1^2}{\left|4(3\alpha'\beta' + 2)^2\left[\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m\right]^2\right|}.$$

References:

[1] S. A. Al - Ameedee , W. G. Atshan and F. A. Al - Maamori , Coefficients estimates of bi - univalent functions defined by new subclass function , Joumal of Physics : Conference Series , 1530 (2020) 012105 , 1-8.

[2] S. A. Al - Ameedee , W. G. Atshan and F. A. Al - Maamori , Second Hankel determinant for certain subclasses of bi - univalent functions , Journal of Physics : Conference Series , 1664 (2020) 012044 , 1-9 .

[3] R. M. Ali , S. K. Lee , V. Ravichandran and S. Subramanian , Coefficient estimates for bi - univalent Ma - Minda starlike and convex functions , Appl. Math. Lett. , 25 (3) (2012) , 344-351 .

[4] O. Altintas and S. Owa , Majorizations and quasi - subordinations for certain analytic functions , Proc. Jpn . Acad. Ser. A , 68 (7) (1992), 181-185.

[5] S. Altinkaya and S. Yalcin , Faber polynomial coefficient bounds for a subclass of bi - univalent functions, C. R. Acad. Sci. Paris , Ser. I. , (2015) (353) :1075-1080.

[6] S. Altinkaya and S. Yalcin , Coefficient bounds for a subclass of bi-univalent , TWMS Journal of Pure and Applied Mathematics, 6 (2015), 180-185 .

[7] S. Altinkaya and S. Yalcin , On a new subclass of bi - univalent functions satisfying subordinate conditions , Acta Universitatis Sapientiae , Mathematica , 7(2015) , 5-14 .

[8] W. G. Atshan , S. Yalcin and R. A. Hadi , Coefficients estimates for special subclasses of k - fold symmetry bi - univalent functions , Mathematics for Applications , 9 (2) (2020) , 83-90.

[9] D. A. Brannan and T. S. Taha. “On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math, 31,(2), (1986), 70-77.

[10] E. Deniz , Certain subclasses of bi - univalent functions satisfying subordinate conditions , J. Classical Ana. , 2(1) (2013), 49-60.

[11] P. L. Duren , Univalent Functions , in: Grunlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlage, New York, Berlin, Heidelberg and Tokyo, (1983).

[12] R- El - Ashwah and S. Kanas , Fekete - Szeg*ö* inequalities for quasi-subordination functions classes of complex order , Kyungpook Mathematical Journal , 55 (2015); 679-688 .

[13] B. A. Frasin and M. K. Aouf , New subclasses of bi - univalent functions , Appl. Math. Lett. , 24 (2011),1569-1573.

[14] S. P. Goyal , O. Singh and R. Mukherjee , Certain results on a subclass of analytic and bi - univalent functions associated with coefficient estimates and quasi-subordination , Palestine Journal of Mathematics, 5 (1) (2016), 79-85 .

[15] S. G. Hamidi , J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi - close - to - convex functions , C. R. Acad. Sci. Paris. Ser. I.,(2014) (352) : 17-20.

[16] S. Kant , Coefficients estimate for certain subclass of bi - univalent functions associated with quasi - subordinations , Journal of Fractional Calculus and Applied , 9 (1) (2018) , 195-203.

[17] M. Lewin, "On a coefficient problem for bi-univalent functions”, Proc. Amer. Math. Soc., 18, (1967), 63-68.

[18] W. C. Ma and D. Minda , A unified treatment of some special classes of univalent functions , In : Proceedings of the conference on complex analysis , ((Tianjin) , 1992)) , Conference proceedings Lecturer Notes Analysis, International Press , Cambridge , Mass , USA , Vol . 1 (1994) , 157-169 .

[19] N. Magesh , V. K. Balaji and J. Yamini , Certain subclasses of bi - starlike and bi - convex functions based on quasi - subordination , Abstract and Applied Analysis , (2016) , Art . ID3102960,6 pages.

[20] G. Murugusundaramoorthy , T. Janani and N. E. Cho, Bi - univalent functions of complex order based on subordinate conditions involving Hurwitz Lerch Zeta function , East Asian Math. J. , 32 (1) (2016) , 47-59.

.

[21] A. Oshah and M. Darus , On certain analytic functions defined by a generalized Ruscheweyh operator , Romai J. , 11 (2015) , 139–144.

[22] T. Panigrahi and R. K. Raina , Fekete - Szego coefficient functional for quasi - subordination class , Afro. Mat. , 10 pages, 2017.

.

[23] Z. Peng , G. Murugusundaramoorthy and T. Janani , Coefficient estimate of bi - univalent functions of complex order associated with the Hohlov operator , Journal of Complex Analysis, 2014

(2014) , Art. ID 693908 , 6 pages.

[24] C. Ramachandran , R. Ambroseprabhu and N. Magesh , Initial Coeff icient estimates for certain subclasses of bi - univalent functions of Ma - Minda type , Applied Mathematical Sciences , 9 (47) (2015) , 2299- 2308 .

[25] M. S. Robertson, "Quasi-subordination and coefficient conjecture", Bull. Amer. Math. Soc., 76, (1970), 1-9.

[26] F. Y. Ren, S. Owa and S. Fukui, Some inequalities on quasi-subordinate functions, Bull. Aust. Math. Soc., 43(2) (1991), 317-324.

[27] S. Yalcin , W. G. Atshan and H. Z. Hassan , Coefficients assessment for certain subclasses of bi - univalent functions related with Quasi-subordination , Publications De L'Institut Mathematique , Nouvelle série , tome 108 (122) (2020), 155-162.

[28] W. G. Atshan and E. I. Badawi , Results on coefficients estimates for subclasses of analytic and bi - univalent functions , Journal of Physics : Conference Series , 1294 (2019) 033025 , 1-10.

[29] W. G. Alshan , I. A. R. Rahman and A. A. Lupas , Some results of new subclasses for bi - univalent functions using quasi - subordination , Symmetry, 13 (9) (2021) , 1653 , 1-12.

[30] W. G. Atshan , E. I. Badawi and H. Ö. Güney, Applications of quasi- subordination of coefficients estimates for new subclasses of bi - univalent functions , Acta Universitaties Apulensis, 68 (2021) , pp. 51-62.

[31] S. D. Theyab , W. G. Atshan and H. K . Abdullah , New results of modern concept on the fourth - Hankel determinant of a certain subclass of analytic functions , Int . J. Nonlinear Anal . Appl., 12 (special Issue) ,Winter and Spring (2021), 2243-2255.

[32] B. K. Mihsin , W. G. Atshan and S. S. Alhily , New results on fourth -order Hankel determinants for convex functions related to the sine function, Int . J. Nonlinear Anal . Appl.,12 (Special Issue) , Winter and Spring (2021) , 2339-2352.