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PD Mean Cordial Labeling for Two Graphs

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ABSTRACT

Let h be a map from $V(G)$ to $\{1,2\}$. We associate two integers $P = h(u)h(v)$ and $D = \left\lfloor\frac{h(u)}{h(v)}\right\rfloor$, where $h(u) \geq h(v)$. For each edge uv assign the label $\left\lceil\frac{P+D}{2}\right\rceil$. Then h is called a PD mean cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i,j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labelled with $x (x = 1,2)$ respectively. A graph G is PD mean cordial if it satisfies PD mean cordial labeling. In this paper, we prove two star, two wheel and two helm are PD mean cordial graph.

Mathematics subject classification: 05C78

Keywords : Mean cordial, Star graph, Wheel graph, Helm graph.

1. INTRODUCTION

One of the fast growing areas of graph theory is graph labeling. Let $G = (V, E)$ be a graph with vertex set V and the edge set E. Labeling of graphs are used in radar, circuit design, communication network etc. In 1987, Cahit introduced the concept of cordial labeling[3]. He restricted the range set of h to $\{0,1\}$. The SD prime cordial labeling was first introduced by G. C. Lau et al. in 2016 [6]. Here we introduce PD mean cordial labeling as a new notation. In any cordial labeling, the values are restricted to 0 and 1. In PD mean cordial labeling of graph, the values assigned to the vertex are restricted to 1 and 2[2]. A graph which admits PD mean cordial labeling is called as PD mean cordial graph.

2. PRELIMINARIES

Definition 2.1

Disconnected graphs are connected by an edge in order to form a connected graph is known as wedge. It is denoted as ^.

Definition 2.2

The graph $W_\alpha = C_\alpha + K_1$ is called wheel, a vertex of degree 3 on the cycle is called a rim vertex. A vertex which is adjacent to all the rim vertices is called the central vertex. The edges with one end incident with a rim vertex and the other incident with the central vertex are called spokes.

Definition 2.3

The helm graph H_α is constructed from the wheel by attaching a pendent edge at each vertex of the α- cycle of the wheel.

3. MAIN RESULTS

Theorem 3.1

The two star $K_{1,\alpha_1} \wedge K_{1,\alpha_2}$ is a PD mean cordial graph.

Proof. Let the graph $G = K_{1,\alpha_1} \wedge K_{1,\alpha_2}$. The vertex and edge set is given by $V(G) = [\{m_\theta, \quad 0 \leq \theta \leq \alpha_1\} \cup \{n_\varphi, 0 \leq \varphi \leq \alpha_2\}]$ and $E(G) = [\{m_0 m_\theta, 1 \leq \theta \leq \alpha_1\} \cup \{m_\theta n_\varphi,$ for any one of $0 \leq \theta \leq \alpha_1$ and $0 \leq \varphi \leq \alpha_2\} \cup \{n_0 n_\varphi, 1 \leq \varphi \leq \alpha_2\}$]. Then G has $\alpha_1 + \alpha_2 + 2$ vertices and $\alpha_1 + \alpha_2 + 1$ edges. To prove $K_{1,\alpha_1} \wedge K_{1,\alpha_2}$ is a PD mean cordial.

The following cases satisfies h is PD mean cordial labeling:

Case 1: $\alpha_1 < \alpha_2$

The required vertex labeling of G is defined as follows:

$h(m_\theta) = 2, 0 \leq \theta \leq \alpha_1$, $h(n_\varphi) = 1, 0 \leq \varphi \leq \left\lceil\frac{\alpha_1+\alpha_2}{2}\right\rceil$,

$h(n_\varphi) = 2, \left\lceil\frac{\alpha_1+\alpha_2+2}{2}\right\rceil \leq \varphi \leq \alpha_2$.

Illustration of the labeling for the two star $K_9 \wedge K_{11}$ is given in Figure. 3.1.

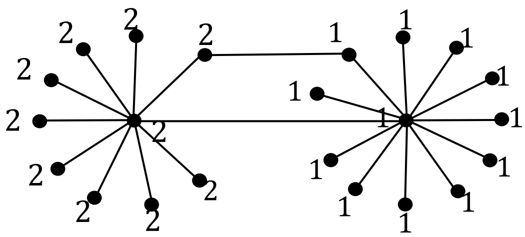

Figure. 3.1

The related edge labeling of G is given below:

$h(m_0 m_\theta) = 2, 1 \leq \theta \leq \alpha_1$, $h(n_0 n_\varphi) = 1$, $1 \leq \varphi \leq \left\lceil\frac{\alpha_1+\alpha_2}{2}\right\rceil$,

$h(n_0 n_\varphi) = 2$, $\left\lceil\frac{\alpha_1+\alpha_2+2}{2}\right\rceil \leq \varphi \leq \alpha_2$.

The wedge labeling of G is $h(m_\theta n_\varphi) = 2$ *for any one of* θ, φ.

Case 2: $\alpha_1 = \alpha_2$

The required vertex labeling of G is defined as follows:

$h(m_\theta) = 2, 0 \leq \theta \leq \alpha_1$, $h(n_\varphi) = 1, 0 \leq \varphi \leq \alpha_2$.

The related edge labeling of G is given below:

$h(m_0 m_\theta) = 2, 1 \leq \theta \leq \alpha_1$, $h(n_0 n_\varphi) = 1, 1 \leq \varphi \leq \alpha_2$.

The wedge labeling of G is $h(m_\theta n_\varphi) = 2 \, for \, any \, one \, of \, \theta, \varphi.$

Case 3: $\alpha_1 > \alpha_2$

The required vertex labeling of G is defined as follows:

$h(m_\theta) = 1, 0 \le \theta \le \left\lceil\frac{\alpha_1+\alpha_2}{2}\right\rceil$, $h(n_\varphi) = 2,\ 0 \le \varphi \le \alpha_2$,

$h(m_\theta) = 2, \left\lceil\frac{\alpha_1+\alpha_2+2}{2}\right\rceil \le \theta \le \alpha_1$.

The related edge labeling of G is given below:

$h(m_0 m_\theta) = 1, 1 \le \theta \le \left\lceil\frac{\alpha_1+\alpha_2}{2}\right\rceil$, $h(n_0 n_\varphi) = 2 \, for \, 1 \le \varphi \le \alpha_2$,

$h(m_0 m_\theta) = 2,\ \left\lceil\frac{\alpha_1+\alpha_2+2}{2}\right\rceil \le \theta \le \alpha_1$.

The wedge labeling of G is $h(m_\theta n_\varphi) = 2$ for any one of θ, φ.

From all the three cases, $v_h(1) = \left\lfloor\frac{\alpha_1+\alpha_2+3}{2}\right\rfloor$, $v_h(2) = \left\lfloor\frac{\alpha_1+\alpha_2+2}{2}\right\rfloor$ and $e_h(1) = \left\lfloor\frac{\alpha_1+\alpha_2+1}{2}\right\rfloor$, $e_h(2) = \left\lfloor\frac{\alpha_1+\alpha_2+2}{2}\right\rfloor$.

Therefore, the graph satisfies the condition $|v_h(i) - v_h(j)| \le 1$ and $|e_h(i) - e_h(j)| \le 1$, $(i,j) \epsilon \{1,2\}$.

Theorem 3.2. Two wheel $W_{\alpha_1}{}^\wedge W_{\alpha_2}$ is a PD mean cordial graph iff $|\alpha_i - \alpha_j| \ne 2n,\quad n \, \epsilon \, N, \alpha_1 = \alpha_2 = 4,5,6,\dots,\ i,j \, \epsilon \, \{1,2\}$.

Proof. Let the graph $= W_{\alpha_1}{}^\wedge W_{\alpha_2}$. The vertex and edge set of G is given by $V(G) = [\{m_\theta,\quad 0 \le \theta \le \alpha_1\} \cup \{n_\varphi, 0 \le \varphi \le \alpha_2\}]$ and $E(G) = [\{m_0 m_\theta, 1 \le \theta \le \alpha_1\} \cup \{n_0 n_\varphi, 1 \le \varphi \le \alpha_2\} \cup \{m_\theta m_{\theta+1}, 1 \le \theta \le \alpha_1 - 1\} \cup \{m_1 m_{\alpha_1}\} \cup \{n_\varphi n_{\varphi+1}, 1 \le \varphi \le \alpha_2 - 1\} \cup \{n_1 n_{\alpha_2}\} \cup \{m_\theta n_\varphi$ for any one of $0 \le \theta \le \alpha_1$ and $0 \le \varphi \le \alpha_2\}]$. Then G has $\alpha_1 + \alpha_2$ vertices and $2(\alpha_1 + \alpha_2 - 2) + 1$ edges. To prove that G is a PD mean cordial for $|\alpha_i - \alpha_j| \ne 2n,\ n \, \epsilon \, N,\ \alpha_1 = \alpha_2 = 4,5,6,\dots,\ i,j \, \epsilon \, \{1,2\}$.

The following cases satisfies the PD mean cordial labeling:

Case 1: $\alpha_2 = \alpha_1 + 1$

Consider the graph $= W_{\alpha_1}{}^\wedge W_{\alpha_2}$, where $\alpha_2 = \alpha_1 + 1$.

The required vertex labeling of G is defined as follows:

Define a map $h: V(G) \to \{1,2\}$ by

$h(m_\theta) = 1, 0 \le \theta \le \alpha_1$, $h(n_0) = 1$, $h(n_\varphi) = 2, 1 \le \varphi \le \alpha_2$.

The related edge labeling of G is given below :

$h(m_0 m_\theta) = 1, 1 \le \theta \le \alpha_1$, $h(n_0 n_\varphi) = 2, 1 \le \varphi \le \alpha_2$,

$h(m_1 m_{\alpha_1}) = 1$, $h(m_\theta m_{\theta+1}) = 1, 1 \le \theta \le \alpha_1 - 1$,

$h(n_1 n_{\alpha_2}) = 2$, $h(n_\varphi n_{\varphi+1}) = 2, 1 \le \varphi \le \alpha_2 - 1$,

The wedge labeling of G is $h(m_\theta n_0) = 1$ for any one of θ.

Case 2: $\alpha_2 = \alpha_1$

Consider the graph $= W_{\alpha_1}{}^\wedge W_{\alpha_2}$, where $\alpha_2 = \alpha_1$.

The required vertex labeling of G is defined as follows:

$h(m_\theta) = 1, 0 \le \theta \le \alpha_1$, $h(n_\varphi) = 2, 0 \le \varphi \le \alpha_2$.

The related edge labeling of G is given below :

$h(m_0 m_\theta) = 1, 1 \le \theta \le \alpha_1$, $h(n_0 n_\varphi) = 2, 1 \le \varphi \le \alpha_2$,

$h(m_1 m_{\alpha_1}) = 1$, $h(m_\theta m_{\theta+1}) = 1, 1 \le \theta \le \alpha_1 - 1$,

$h(n_1 n_{\alpha_2}) = 2$, $h(n_\varphi n_{\varphi+1}) = 2, 1 \le \varphi \le \alpha_2 - 1$.

The wedge labeling of G is $h(m_\theta n_\varphi) = 2$ for any one of $0 \le \theta \le \alpha_1$ and $0 \le \varphi \le \alpha_2$.

Case 3: $\alpha_2 = \alpha_1 - 1$

Consider the graph $= W_{\alpha_1}{}^\wedge W_{\alpha_2}$, where $\alpha_2 = \alpha_1 - 1$.

The required vertex labeling of G is defined as follows:

$h(m_\theta) = 1, 0 \le \theta \le \alpha_1$, $h(n_\varphi) = 2, 0 \le \varphi \le \alpha_2$.

The related edge labeling of G is given below:

$h(m_0 m_\theta) = 1, 1 \le \theta \le \alpha_1$, $h(m_\theta m_{\theta+1}) = 1, 1 \le \theta \le \alpha_1 - 1$,

$h(m_1 m_{\alpha_1}) = 1$, $h(n_0 n_\varphi) = 2, 1 \le \varphi \le \alpha_2$,

$h(n_1 n_{\alpha_2}) = 2$, $h(n_\varphi n_{\varphi+1}) = 2, 1 \le \varphi \le \alpha_2 - 1$.

The wedge labeling of G is $h(m_\theta n_\varphi) = 2$ for any one of $0 \le \theta \le \alpha_1$ and $0 \le \varphi \le \alpha_2$.

Case 4: $\alpha_2 - \alpha_1 = 2n + 1, n \, \epsilon \, N$.

Consider the graph $= W_{\alpha_1}{}^\wedge W_{\alpha_2}$, where $\alpha_2 - \alpha_1 = 2n + 1, n \, \epsilon \, N$.

The required vertex labeling of G is defined as follows:

$h(m_\theta) = 2, 0 \le \theta \le \alpha_1$, $h(n_\varphi) = 2, 1 \le \varphi \le \frac{\alpha_2-\alpha_1-1}{2}$,

$h(n_0) = 1$, $h(n_\varphi) = 1$, $\frac{\alpha_2-\alpha_1+1}{2} \le \varphi \le \alpha_2$.

The related edge labeling of G is given below:

$h(m_0 m_\theta) = 2, 1 \le \theta \le \alpha_1$, $h(m_\theta m_{\theta+1}) = 2, 1 \le \theta \le \alpha_1 - 1$,

$h(m_1 m_{\alpha_1}) = 2$, $h(n_\varphi n_{\varphi+1}) = 2, 1 \le \varphi \le \frac{\alpha_2-\alpha_1-1}{2}$,

$h(n_0 n_\varphi) = 2, 1 \le \varphi \le \frac{\alpha_2-\alpha_1-1}{2}$, $h(n_0 n_\varphi) = 1, \frac{\alpha_2-\alpha_1+1}{2} \le \varphi \le \alpha_2$,

$h(n_1 n_{\alpha_2}) = 2$, $h(n_\varphi n_{\varphi+1}) = 1, \frac{\alpha_2-\alpha_1+1}{2} \le \varphi \le \alpha_2 - 1$.

The wedge labeling of G is $h(m_\theta n_\varphi) = 2$ for any one of $0 \le \theta \le \alpha_1$ and $0 \le \varphi \le \alpha_2$.

Case 5: $\alpha_1 - \alpha_2 = 2n + 1, n \, \epsilon \, N$.

Consider the graph $= W_{\alpha_1}{}^\wedge W_{\alpha_2}$, where $\alpha_1 - \alpha_2 = 2n + 1, n \, \epsilon \, N$.

The required vertex labeling of G is defined as follows:

$h(m_0) = 1$, $h(m_\theta) = 2, 1 \le \theta \le \frac{\alpha_1-\alpha_2-1}{2}$,

$h(m_\theta) = 1, \frac{\alpha_1-\alpha_2+1}{2} \le \theta \le \alpha_1$, $h(n_\varphi) = 2, 0 \le \varphi \le \alpha_2$.

The related edge labeling of G is given below :

$h(m_0 m_\theta) = 2, 1 \le \theta \le \frac{\alpha_1-\alpha_2-1}{2}$, $h(m_0 m_\theta) = 1, \frac{\alpha_1-\alpha_2+1}{2} \le \theta \le \alpha_1$,

$h(m_1 m_{\alpha_1}) = 2$, $h(m_\theta m_{\theta+1}) = 2, 1 \le \theta \le \frac{\alpha_1-\alpha_2-1}{2}$,

$h(n_1 n_{\alpha_2}) = 2$, $h(m_\theta m_{\theta+1}) = 1$, $\frac{\alpha_1-\alpha_2+1}{2} \le \theta \le \alpha_1 - 1$,

$h(n_0 n_\varphi) = 2, 1 \le \varphi \le \alpha_2$, $h(n_\varphi n_{\varphi+1}) = 2, 1 \le \varphi \le \alpha_2 - 1$.

The wedge labeling of G is $h(m_\theta n_\varphi) = 2$ for any one of $0 \le \theta \le \alpha_1$ and $0 \le \varphi \le \alpha_2$.

The above cases are PD mean cordial labeling as shown in the following table 3.1 and 3.2 :

Nature of α_2	$v_h(1)$	$v_h(2)$
$\alpha_2 = \alpha_1 + 1$	$\alpha_1 + 2$	$\alpha_1 + 1$
$\alpha_2 = \alpha_1$	α_1	α_1
$\alpha_2 = \alpha_1 - 1$	$\alpha_1 + 1$	α_1
$\alpha_2 - \alpha_1 = 2n + 1, n \, \epsilon \, N$	$\frac{\alpha_1 + \alpha_2 + 3}{2}$	$\frac{\alpha_1 + \alpha_2 + 1}{2}$
$\alpha_1 - \alpha_2 = 2n + 1, n \, \epsilon \, N$	$\frac{\alpha_1 + \alpha_2 + 3}{2}$	$\frac{\alpha_1 + \alpha_2 + 1}{2}$

Table 3.1

Nature of α_2	$e_h(1)$	$e_h(2)$
$\alpha_2 = \alpha_1 + 1$	$2\alpha_1 + 1$	$2\alpha_1 + 2$
$\alpha_2 = \alpha_1$	$2\alpha_1$	$2\alpha_1 + 1$
$\alpha_2 = \alpha_1 - 1$	$2\alpha_1$	$2\alpha_1 - 1$
$\alpha_2 - \alpha_1 = 2n + 1, n \, \epsilon \, N$	$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2 + 1$
$\alpha_1 - \alpha_2 = 2n + 1, n \, \epsilon \, N$	$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2 + 1$

Table 3.2

Hence, G is a PD mean cordial graph for $|\alpha_i - \alpha_j| \neq 2n$, $n \in N, \alpha_1 = \alpha_2 = 4, 5, 6, \ldots$, $i, j \in \{1, 2\}$.
Conversly, suppose $|\alpha_i - \alpha_j| = 2n$, $n \in N, \alpha_1 = \alpha_2 = 3, 4, 5, 6, \ldots$, $i, j \in \{1, 2\}$ and h is a PD mean cordial. Here $|V(G)| = \alpha_1 + \alpha_2 + 2$ and $|E(G)| = 2(\alpha_1 + \alpha_2) + 1$. If $v_f(i) = \frac{\alpha_1+\alpha_2+2}{2}$, $i \in \{1,2\}$ then we must have $e_f(i) = \alpha_1 + \alpha_2$ or $\alpha_1 + \alpha_2 + 1, i \in \{1,2\}$. But we get $e_f(1) \leq \alpha_1 + \alpha_2 - 1$ and $e_f(2) \geq \alpha_1 + \alpha_2 + 2$. From this $|v_f(i) - v_f(j)| \leq 1$ but $|e_f(i) - e_f(j)| \geq 3, (i, j) \in \{1,2\}$, a contradiction. If $v_f(i) \neq \frac{\alpha_1+\alpha_2+2}{2}, i \in \{1,2\}$ then $|v_f(i) - v_f(j)| > 1, (i, j) \in \{1,2\}$, a contradiction.

Illustration of the labeling for the two wheel W_7^W_4 is given in Figure. 3.2.

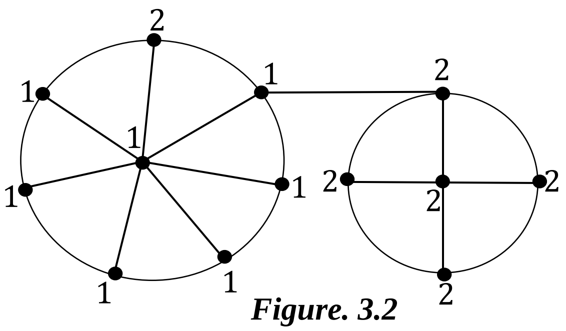

Figure. 3.2

Theorem 3.3. Two helm graph H_{α_1}^H_{α_2} is a PD mean cordial graph.
Proof. Let the graph $= H_{\alpha_1}$^H_{α_2}. The vertex and edge set of G is given by $V(G) = [\{m_\theta, \quad 0 \leq \theta \leq \alpha_1\} \cup \{m'_\theta, 0 \leq \theta \leq \alpha_2\} \cup \{n_\varphi, 1 \leq \varphi \leq \alpha_1\} \cup \{n'_\varphi, 1 \leq \varphi \leq \alpha_2\}]$.
$E(G) = [\{m_0 m_\theta, 1 \leq \theta \leq \alpha_1\} \cup \{m_\theta m_{\theta+1}, 1 \leq \theta \leq \alpha_1 - 1\} \cup \{m_\theta n_\varphi, 1 \leq \theta, \varphi \leq \alpha_1\} \cup \{m_1 m_{\alpha_1}\} \cup \{m'_0 m'_\theta, 1 \leq \theta \leq \alpha_2\} \cup \{m'_\theta m'_{\theta+1}, 1 \leq \theta \leq \alpha_2 - 1\} \cup \{m'_1 m'_{\alpha_2}\} \cup \{m'_\theta n'_\varphi, 1 \leq \theta, \varphi \leq \alpha_2\} \cup \{m_\theta m'_\theta \; or \; m_\theta n'_\varphi \; or \; n_\varphi m'_\theta \; or \; n_\varphi n'_\varphi \; for \; any \; one \; of \; \theta, \varphi\}]$. Then G has $2(\alpha_1 + \alpha_2 + 1)$ vertices and $3(\alpha_1 + \alpha_2) + 1$ edges. To prove that G is a PD mean cordial for $\alpha_1 \leq \alpha_2$. Define a map $h: V(G) \to \{1, 2\}$ and $h: E(G) \to \{1, 2\}$. The following cases satisfies the PD mean cordial labeling:
Case 1: $\alpha_1 \leq \alpha_2$
Subcase 1: $\alpha_2 = \alpha_1$
Consider the graph $= H_{\alpha_1}$^H_{α_2}, where $\alpha_2 = \alpha_1$.
The required vertex labeling of G is defined as follows:
$h(m_\theta) = 2, 0 \leq \theta \leq \alpha_1, h(m'_\theta) = 1, 0 \leq \theta \leq \alpha_2$,
$h(n_{\alpha_1}) = 1, h(n_\varphi) = 2, 1 \leq \varphi \leq \alpha_1 - 1$,
$h(n'_1) = 2, h(n'_\varphi) = 1, 2 \leq \varphi \leq \alpha_2$.
The related edge labeling of G is given below:
$h(m_0 m_\theta) = 2, 1 \leq \theta \leq \alpha_1, h(m_\theta n_\varphi) = 2, 1 \leq \theta, \varphi \leq \alpha_1$,
$h(m_1 m_{\alpha_1}) = 2, h(m_\theta m_{\theta+1}) = 2, 1 \leq \theta \leq \alpha_1 - 1$,
$h(m'_0 m'_\theta) = 1, 1 \leq \theta \leq \alpha_2, \quad h(m'_\theta m'_{\theta+1}) = 1, 1 \leq \theta \leq \alpha_2 - 1$,
$h(m'_1 m'_{\alpha_2}) = 1, \quad h(m'_1 n'_1) = 2, \quad h(m'_\theta n'_\varphi) = 1, 2 \leq \theta, \varphi \leq \alpha_2$.
The wedge labeling of G is given below:
$h(n_{\alpha_1} m'_\theta) = 1 \; for \; any \; one \; of \; \theta$ or $h(n_{\alpha_1} n'_\varphi) = 1 \; for \; any \; one \; of \; 2 \leq \varphi \leq \alpha_2$.

Subcase 2: $\alpha_2 = \alpha_1 + 1$
Consider the graph $= H_{\alpha_1}$^H_{α_2}, where $\alpha_2 = \alpha_1 + 1$.
The required vertex labeling of G is defined as follows:
$h(m_\theta) = 2, 0 \leq \theta \leq \alpha_1, h(m'_\theta) = 1, 0 \leq \theta \leq \alpha_2$,
$h(n_{\alpha_1}) = 1, h(n_\varphi) = 2, 1 \leq \varphi \leq \alpha_1 - 1$,
$h(n'_\varphi) = 2, 1 \leq \varphi \leq 2, h(n'_\varphi) = 1, 3 \leq \varphi \leq \alpha_2$.
The related edge labeling of G is given below:
$h(m_0 m_\theta) = 2, 1 \leq \theta \leq \alpha_1, h(m_\theta n_\varphi) = 2, 1 \leq \theta, \varphi \leq \alpha_1$,
$h(m_\theta m_{\theta+1}) = 2, 1 \leq \theta \leq \alpha_1 - 1, h(m'_0 m'_\theta) = 1, 1 \leq \theta \leq \alpha_2$,
$h(m_1 m_{\alpha_1}) = 2, \quad h(m'_1 m'_{\alpha_2}) = 1, \quad h(m'_\theta m'_{\theta+1}) = 1, 1 \leq \theta \leq \alpha_2 - 1$,
$h(m'_\theta n'_\varphi) = 2, \; 1 \leq \theta, \varphi \leq 2, \; h(m'_\theta n'_\varphi) = 1, 3 \leq \theta, \varphi \leq \alpha_2$.
The wedge labeling of G is given below:
$h(n_{\alpha_1} m'_\theta) = 1 \; for \; any \; one \; of \; \theta$ or $h(n_{\alpha_1} n'_\varphi) = 1 \; for \; any \; one \; of \; 3 \leq \varphi \leq \alpha_2$.
Subcase 3: $\alpha_2 = \alpha_1 + 2$
Consider the graph $= H_{\alpha_1}$^H_{α_2}, where $\alpha_1 + 2 = \alpha_2$.
The required vertex labeling of G is defined as follows:
$h(m_\theta) = 2, 0 \leq \theta \leq \alpha_1, h(m'_\theta) = 1, 0 \leq \theta \leq \alpha_2$,
$h(n_{\alpha_1}) = 1, h(n_\varphi) = 2, 1 \leq \varphi \leq \alpha_1 - 1$,
$h(n'_\varphi) = 2, 1 \leq \varphi \leq 3, h(n'_\varphi) = 1, 4 \leq \varphi \leq \alpha_2$.
The related edge labeling of G is given below:
$h(m_0 m_\theta) = 2, 1 \leq \theta \leq \alpha_1, \; h(m_\theta m_{\theta+1}) = 2, 1 \leq \theta \leq \alpha_1 - 1$,
$h(m_\theta n_\varphi) = 2, 1 \leq \theta, \varphi \leq \alpha_1, h(m'_0 m'_\theta) = 1, 1 \leq \theta \leq \alpha_2$,
$h(m_1 m_{\alpha_1}) = 2, \quad h(m'_1 m'_{\alpha_2}) = 1, \quad h(m'_\theta m'_{\theta+1}) = 1, 1 \leq \theta \leq \alpha_2 - 1$,
$h(m'_\theta n'_\varphi) = 2, \; 1 \leq \theta, \varphi \leq 3, \; h(m'_\theta n'_\varphi) = 1, 4 \leq \theta, \varphi \leq \alpha_2$.
The wedge labeling of G is given below:
$h(n_{\alpha_1} m'_\theta) = 1 \; for \; any \; one \; of \; \theta$ or $h(n_{\alpha_1} n'_\varphi) = 1 \; for \; any \; one \; of \; 4 \leq \varphi \leq \alpha_2$.
Subcase 4: $\alpha_2 = \alpha_1 + 3$
Consider the graph $= H_{\alpha_1}$^H_{α_2}, where $\alpha_1 + 3 = \alpha_2$.
The required vertex labeling of G is defined as follows:
$h(m_\theta) = 2, 0 \leq \theta \leq \alpha_1, h(m'_\theta) = 1, \; 0 \leq \theta \leq \alpha_2$,
$h(n_{\alpha_1}) = 1, h(n_\varphi) = 2, 1 \leq \varphi \leq \alpha_1 - 1$,
$h(n'_\varphi) = 2, 1 \leq \varphi \leq 4, h(n'_\varphi) = 1, 5 \leq \varphi \leq \alpha_2$.
The related edge labeling of G is given below:
$h(m_0 m_\theta) = 2, 1 \leq \theta \leq \alpha_1, h(m_\theta m_{\theta+1}) = 2, 1 \leq \theta \leq \alpha_1 - 1$,
$h(m_1 m_{\alpha_1}) = 2, h(m'_1 m'_{\alpha_2}) = 1, h(m_\theta n_\varphi) = 2, 1 \leq \theta, \varphi \leq \alpha_1$,
$h(m'_0 m'_\theta) = 1, 1 \leq \theta \leq \alpha_2, \quad h(m'_\theta m'_{\theta+1}) = 1, 1 \leq \theta \leq \alpha_2 - 1$,
$h(m'_\theta n'_\varphi) = 2, \; 1 \leq \theta, \varphi \leq 4, \; h(m'_\theta n'_\varphi) = 1, 5 \leq \theta, \varphi \leq \alpha_2$.
The wedge labeling of G is given below:
$h(n_{\alpha_1} n'_\varphi) = 2 \; for \; any \; one \; of \; 1 \leq \varphi \leq 4$.
Subcase 5: $\alpha_2 = \alpha_1 + 4$
Consider the graph $= H_{\alpha_1}$^H_{α_2}, where $\alpha_1 + 4 = \alpha_2$.
The required vertex labeling of G is defined as follows:
$h(m_\theta) = 2, 0 \leq \theta \leq \alpha_1, h(m'_\theta) = 1, 0 \leq \theta \leq \alpha_2$,
$h(n_{\alpha_1}) = 1, h(n_\varphi) = 2,1 \leq \varphi \leq \alpha_1 - 1$,
$h(n'_\varphi) = 2, 1 \leq \varphi \leq 5, h(n'_\varphi) = 1, 6 \leq \varphi \leq \alpha_2$.
The related edge labeling of G is given below:

$h(m_0m_\theta) = 2, 1 \le \theta \le \alpha_1, h(m_\theta n_\varphi) = 2, 1 \le \theta, \varphi \le \alpha_1,$
$h(m_1m_{\alpha_1}) = 2,\ h(m'_1m'_{\alpha_2}) = 1,\ h(m'_0m'_\theta) = 1, 1 \le \theta \le \alpha_2,$
$h(m_\theta m_{\theta+1}) = 2, 1 \le \theta \le \alpha_1 - 1, \quad h(m'_\theta m'_{\theta+1}) = 1, 1 \le \theta \le \alpha_2 - 1,$
$h(m'_\theta n'_\varphi) = 2, 1 \le \theta, \varphi \le 5, \quad h(m'_\theta n'_\varphi) = 1, 6 \le \theta, \varphi \le \alpha_2.$
The wedge labeling of G is given below:
$h(n_{\alpha_1}n'_\varphi) = 2\ for\ any\ one\ of\ 1 \le \varphi \le 5.$

Subcase 6: $\alpha_2 = \alpha_1 + 2n + 1, n\epsilon\, N - \{1\}$
Consider the graph $= H_{\alpha_1}{}^\wedge H_{\alpha_2}$, where $\alpha_1 + 2n + 1 = \alpha_2, n\epsilon\, N - \{1\}$.
The required vertex labeling of G is defined as follows:
$h(m'_0) = 1, h(m_\theta) = 2, 0 \le \theta \le \alpha_1,$
$h(n_{\alpha_1}) = 1, h(n_\varphi) = 2, 1 \le \varphi \le \alpha_1 - 1,$
$h(m'_\theta) = 2, 1 \le \theta \le \frac{\alpha_2-\alpha_1-3}{2}, h(m'_\theta) = 1, \frac{\alpha_2-\alpha_1-1}{2} \le \theta \le \alpha_2,$
$h(n'_\varphi) = 2, 1 \le \varphi \le \frac{\alpha_2-\alpha_1+5}{2}, \quad h(n'_\varphi) = 1, \frac{\alpha_2-\alpha_1+7}{2} \le \varphi \le \alpha_2.$
The related edge labeling of G is given below:
$h(m_0m_\theta) = 2, 1 \le \theta \le \alpha_1, h(m_\theta m_{\theta+1}) = 2, 1 \le \theta \le \alpha_1 - 1,$
$h(m'_1m'_{\alpha_2}) = 2, h(m_1m_{\alpha_1}) = 2, h(m_\theta n_\varphi) = 2, 1 \le \theta, \varphi \le \alpha_1,$
$h(m'_0m'_\theta) = 2, 1 \le \theta \le \frac{\alpha_2-\alpha_1-3}{2}, \quad h(m'_0m'_\theta) = 1, \frac{\alpha_2-\alpha_1-1}{2} \le \theta \le \alpha_2,$
$h(m'_\theta m'_{\theta+1}) = 2, \quad 1 \le \theta \le \frac{\alpha_2-\alpha_1-3}{2}, \quad h(m'_\theta m'_{\theta+1}) = \ 1\ \frac{\alpha_2-\alpha_1-1}{2} \le \theta \le \alpha_2 - 1,$
$h(m'_\theta n'_\varphi) = 2, 1 \le \theta, \varphi \le \frac{\alpha_2-\alpha_1+5}{2}, \quad h(m'_\theta n'_\varphi) = 1, \frac{\alpha_2-\alpha_1+7}{2} \le \theta, \varphi \le \alpha_2.$
The wedge labeling of G is given below:
$h(n_{\alpha_1}m'_\theta) = 1\ for\ any\ one\ of\ \frac{\alpha_2-\alpha_1-1}{2} \le \theta \le \alpha_2$ or
$h(n_{\alpha_1}m'_0) = 1$ or
$h(n_{\alpha_1}n'_\varphi) = 1\ \ for\ any\ one\ of\ \frac{\alpha_2-\alpha_1+7}{2} \le \varphi \le \alpha_2.$

Subcase 7: $\alpha_2 = \alpha_1 + 2n, n\epsilon\, N - \{1, 2\}$
Consider the graph $= H_{\alpha_1}{}^\wedge H_{\alpha_2}$, where $\alpha_1 + 2n = \alpha_2, n\,\epsilon\, N - \{1, 2\}$.
The required vertex labeling of G is defined as follows:
$h(m'_0) = 1,\ h(m_\theta) = 2, 0 \le \theta \le \alpha_1,\ h(n_{\alpha_1}) = 1,\ h(n_\varphi) = 2, 1 \le \varphi \le \alpha_1 - 1,$
$h(m'_\theta) = 2, 1 \le \theta \le \frac{\alpha_2-\alpha_1-4}{2}, h(m'_\theta) = 1, \frac{\alpha_2-\alpha_1-2}{2} \le \theta \le \alpha_2,$
$h(n'_\varphi) = 2, 1 \le \varphi \le \frac{\alpha_2-\alpha_1+6}{2}, \quad h(n'_\varphi) = 1, \frac{\alpha_2-\alpha_1+8}{2} \le \varphi \le \alpha_2.$
The related edge labeling of G is given below:
$h(m_0m_\theta) = 2, 1 \le \theta \le \alpha_1,\ h(m_\theta m_{\theta+1}) = 2, 1 \le \theta \le \alpha_1 - 1,$
$h(m_1m_{\alpha_1}) = 2, h(m'_1m'_{\alpha_2}) = 2,\ h(m_\theta n_\varphi) = 2, 1 \le \theta, \varphi \le \alpha_1,$
$h(m'_0m'_\theta) = 2, 1 \le \theta \le \frac{\alpha_2-\alpha_1-4}{2}, \quad h(m'_0m'_\theta) = 1, \frac{\alpha_2-\alpha_1-2}{2} \le \theta \le \alpha_2,$
$h(m'_\theta m'_{\theta+1}) = 2, 1 \le \theta \le \frac{\alpha_2-\alpha_1-4}{2}, \quad h(m'_\theta m'_{\theta+1}) = 1, \frac{\alpha_2-\alpha_1-2}{2} \le \theta \le \alpha_2 - 1,$
$h(m'_\theta n'_\varphi) = 2, 1 \le \theta, \varphi \le \frac{\alpha_2-\alpha_1+6}{2}, \quad h(m'_\theta n'_\varphi) = 1, \frac{\alpha_2-\alpha_1+8}{2} \le \theta,\ \varphi \le \alpha_2.$
The wedge labeling of G is given below:
$h(n_{\alpha_1}m'_\theta) = 1\ for\ any\ one\ of\ \frac{\alpha_2-\alpha_1-2}{2} \le \theta \le \alpha_2$ or
$h(n_{\alpha_1}m'_0) = 1$ or
$h(n_{\alpha_1}n'_\varphi) = 1\ for\ any\ one\ of\ \frac{\alpha_2-\alpha_1+8}{2} \le \varphi \le \alpha_2.$

The above cases are PD mean cordial labeling as shown in the table 4.3 and 4.4:

Nature of α_2	$v_h(1)$	$v_h(2)$
$\alpha_2 = \alpha_1$	$\alpha_1 + \alpha_2 + 1$	$\alpha_1 + \alpha_2 + 1$
$\alpha_2 = \alpha_1 + 1$	$\alpha_1 + \alpha_2 + 1$	$\alpha_1 + \alpha_2 + 1$
$\alpha_2 = \alpha_1 + 2$	$\alpha_1 + \alpha_2 + 1$	$\alpha_1 + \alpha_2 + 1$
$\alpha_2 = \alpha_1 + 3$	$\alpha_1 + \alpha_2 + 1$	$\alpha_1 + \alpha_2 + 1$
$\alpha_2 = \alpha_1 + 4$	$\alpha_1 + \alpha_2 + 1$	$\alpha_1 + \alpha_2 + 1$
$\alpha_1 + 2n + 1 = \alpha_2, n\epsilon\, N - \{1\}$	$\alpha_1 + \alpha_2 + 1$	$\alpha_1 + \alpha_2 + 1$
$\alpha_1 + 2n = \alpha_2, n\epsilon\, N - \{1, 2\}$	$\alpha_1 + \alpha_2 + 1$	$\alpha_1 + \alpha_2 + 1$

Table 4.3

Nature of α_2	$e_h(1)$	$e_h(2)$
$\alpha_2 = \alpha_1$	$3\alpha_2$	$3\alpha_2 + 1$
$\alpha_2 = \alpha_1 + 1$	$3\alpha_2 - 1$	$3\alpha_2 - 1$
$\alpha_2 = \alpha_1 + 2$	$3\alpha_2 - 2$	$3\alpha_2 - 3$
$\alpha_2 = \alpha_1 + 3$	$3\alpha_2 - 4$	$3\alpha_2 - 4$
$\alpha_2 = \alpha_1 + 4$	$3\alpha_2 - 5$	$3\alpha_2 - 6$
$\alpha_1 + 2n + 1 = \alpha_2, n\epsilon\, N - \{1\}$	$\frac{3\alpha_2 + 3\alpha_1 + 1}{2}$	$\frac{3\alpha_2 + 3\alpha_1 + 1}{2}$
$\alpha_1 + 2n = \alpha_2, n\epsilon\, N - \{1, 2\}$	$\frac{3\alpha_2 + 3\alpha_1 + 2}{2}$	$\frac{3\alpha_2 + 3\alpha_1}{2}$

Table 4.4

Hence, G is a PD mean cordial graph if $\alpha_1 \le \alpha_2$.

Case 2: $\alpha_1 \ge \alpha_2$
In case 1, put $\alpha_1 = \alpha_2$, $\alpha_2 = \alpha_1$, $m = m'$, $m' = m$, $n = n'$, $n' = n$, where $\alpha_1 \ge \alpha_2$.
Hence, the two helm is a PD mean cordial graph for all α_1, α_2.

Illustration of the labeling for the two helm $H_3{}^\wedge H_8$ is given in Figure. 3.3

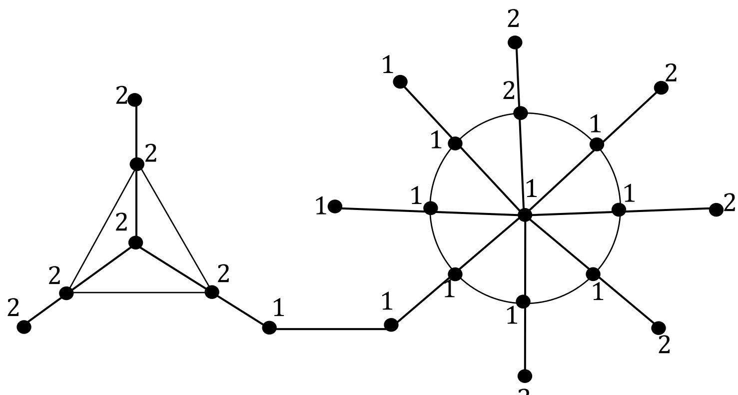

Figure. 3.3

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