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Recurrence of Generalized limiting ratio relations

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ABSTRACT

Progression of a distinct behavioral examination is studied using continued process. Continued process is mostly multidimensional array to previous terms using recursion, in this continued process with the n^{th} term is straight line. Among them, in a sequence or a multidimensional array to previous terms using recursion with constant coefficients and linear recurrences with polynomial coefficients are very much important. Continued process results are generally straight-line recurrences, in this n^{th} term would be straight line regarding to its previous terms. Objective of this paper is finding the fraction of $(l+1)^{th}$ f term to l^{th} term and it's known as mean ratio. Continued process is applied step-by-step for a common sequence to determining a précised iterative relation and mean ratio. By characterizing graphs, we can easily understand exact results obtained which is nearest to 2 on either side of progression. Closest point of progression when, m=18 the result obtained is 1.9999 and then when, m=19 the result obtained is 2.0001.

KEYWORDS: Recurrence relation, limiting ratio, Convergence, Fibonacci sequence.

INTRODUCTION

Continued process with common conditions can be achieved by the Fibonacci sequence way. A Book is published in the title "Libel' Abaci" by a famous Italian Leonardo Fibonacci. In this paper, with respect to the limiting ratios we derived new results to generalize the continued process using Fibonacci sequence. The results obtained are illustrated with some figures. The final result will provide a new insight in understanding the behavior of limiting ratio of generalized continued process.

DEFINITIONS

The recurrence relation of Fibonacci sequence is given by

$$s(l+2)=s(l+1)+s(l), l\geq 0, s(0)=1, s(1)=1$$

(1.1).

In (1.1), we observe that except the first two terms, each term of the sequence is sum of the two previous terms. The generalized recurrence relation of Fibonacci type sequence is defined by

$$s(l+m)=s(l+m-1)+s(l+m-2)+.....+s(l+1)+s(l), l\geq$$

(1.2),

Where

$$s(0)=s(1)=s(2)=.....=s(m-2)=s(m-1)=1.$$

In (1.2), we observe that except for the first m terms, each term is sum of the previous m terms of the sequence.

The ratio of $(l+1)^{th}$ term to the l^{th} term of a sequence as $l\to\infty$ is defined as the limiting ratio of the sequence. We denote the limiting ratio by λ .

Thus, $\lambda=\dfrac{s(l+1)}{s(n)}$ as $l\to\infty$ (1.3)

If λ is the limiting ratio, then for any integer $r\geq 1$ and as $l\to\infty$ we have

$$\frac{s(l+r)}{s(l)}=\frac{s(l+r)}{s(l+r-1)}\times\frac{s(l+r-1)}{s(l+r-2)}\times.......\times\frac{s(l+2)}{s(l+1)}\times\frac{s(l+1)}{s(l)}=\lambda\times\lambda\times.........\times\lambda\times\lambda=\lambda^r$$

(1.4)

SPECIAL CASES

When $m=11$

If $m=11$,then from (1.2), we get

$$s(l+11)=s(l+10)+s(l+9)+s(l+8)+.....+s(l+1)+s(l), l\geq 0, s(0)=s(1)=.....=s(10)=1$$

(2.1)

The characteristic equation of (2.1) is given by

$$t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1=0$$

(2.2)

By Iteration method, we see that the real root of the polynomial in (2.2) is 1.99951 approximately (2.3). Figure 1 verifies this fact.

From (2.1) we get

$$\frac{s(l+11)}{s(l)}=\frac{s(l+10)}{s(l)}+\frac{s(l+9)}{s(l)}+.......+\frac{s(l+2)}{s(l)}+\frac{s(l+1)}{s(l)}+1$$

If λ is the limiting ratio of (2.1), then as $l\to\infty$ from (1.4), we get

$$\lambda^{11}-\lambda^{10}-\lambda^{9}-\lambda^{8}-\lambda^{7}-\lambda^{6}-\lambda^{5}-\lambda^{4}-\lambda^{3}-\lambda^{2}-\lambda-1=0.$$

(2.4)

Hence, the limiting ratio of (2.1) is the positive real root of (2.2) which is 1.99951 approximately. (2.3)

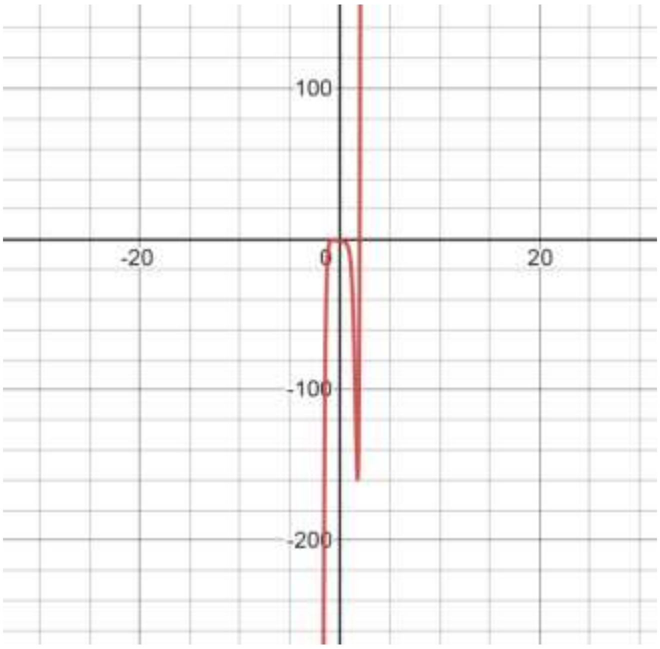

Figure 1: Graph of y = $t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1$

When $m=12$

If $m=12$,then from (1.2), we get

$$s(l+12)=s(l+11)+s(l+10)+s(l+9)+.....+s(l+1)+s(l), l\geq 0, s(0)=s(1)=.....=s(11)=1$$

(2.5)

The characteristic equation of (2.5) is given by

$$t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1=0$$

(2.6)

By Iteration method, we see that the real root of the polynomial in (2.6) is 1.99967approximately (2.7). Figure 2 verifies this fact.

From (2.5) we get

$$\frac{s(l+12)}{s(l)}=\frac{s(l+11)}{s(l)}+\frac{s(l+10)}{s(l)}+.......+\frac{s(l+2)}{s(l)}+\frac{s(l+1)}{s(l)}+1$$

If λ is the limiting ratio of (2.5), then as $l\to\infty$ from (1.4), we get

$$\lambda^{12}-\lambda^{11}-\lambda^{10}-\lambda^{9}-\lambda^{8}-\lambda^{7}-\lambda^{6}-\lambda^{5}-\lambda^{4}-\lambda^{3}-\lambda^{2}-\lambda-1=0$$

. (2.8)

Hence, the limiting ratio of (2.5) is the positive real root of (2.6)Which is 1.99967approximately. (2.7).

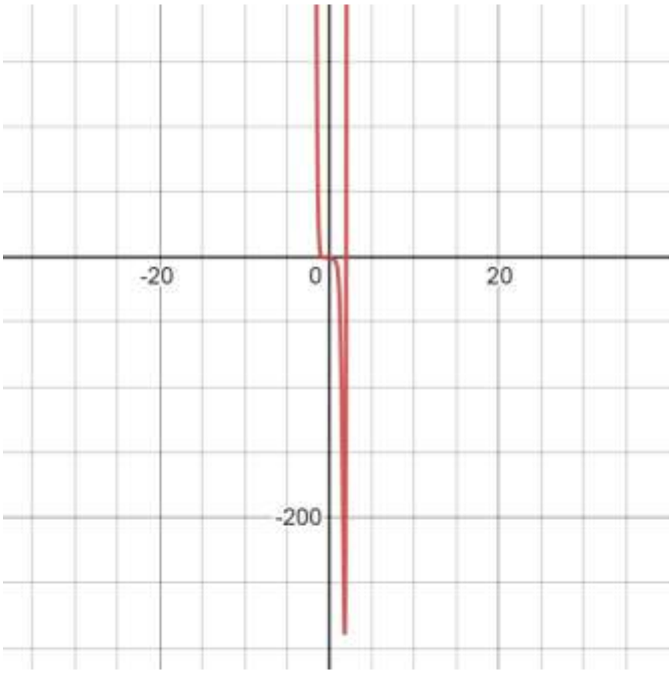

Figure 2:

When $m=13$

Figure 2: Graph of y = $t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1$

When $m=13$

If $m=13$,then from (1.2), we get

$$s(l+13)=s(l+12)+s(l+11)+s(l+10)+.....+s(l+1)+s(l), l\geq 0, s(0)=s(1)=.....=s(12)=1$$

(2.9)

The characteristic equation of (2.9) is given by

$$t^{13}-t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1=0$$

(3.0)

By Iteration method, we see that the real root of the polynomial in (3.0) is1.99988 (3.1)approximately. Figure 3verifies this fact.

From (2.9) we get

$$\frac{s(l+13)}{s(l)}=\frac{s(l+12)}{s(l)}+\frac{s(l+11)}{s(l)}+.......+\frac{s(l+2)}{s(l)}+\frac{s(l+1)}{s(l)}+1$$

If λ is the limiting ratio of (2.9) then as $l\to\infty$ from (1.4), we get

$$\lambda^{13}-\lambda^{12}-\lambda^{11}-\lambda^{10}-\lambda^{9}-\lambda^{8}-\lambda^{7}-\lambda^{6}-\lambda^{5}-\lambda^{4}-\lambda^{3}-\lambda^{2}-\lambda-1=0$$

. (3.2)

Hence, the limiting ratio of (2.9) is the positive real root of (3.0) which is 1.99988approximately. (3.1)

When $m=13$

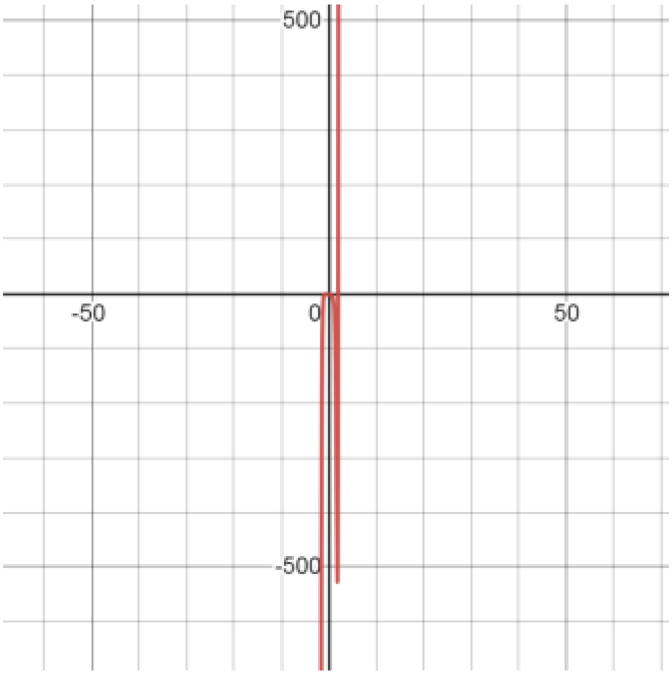

Figure 3: Graph of y = $t^{13}-t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1$

When $m=15$

If $m=15$,then from (1.2), we get

$$s(l+15)=s(l+14)+s(l+13)+s(l+12)+.....+s(l+1)+s(l),l\geq 0,s(0)=s(1)=....=s(14)=1$$

(3.3)

The characteristic equation of(3.3) is given by

$$t^{15}-t^{14}-t^{13}-t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1=0$$

(3.4)

By Iteration method, we see that the positive real root of the polynomial in (3.4) is 1.99997approximately(3.5). Figure 4 verifies this fact.

From (3.3) we get

$$\frac{s(l+15)}{s(l)}=\frac{s(l+14)}{s(l)}+\frac{s(l+13)}{s(l)}+.......+\frac{s(l+2)}{s(l)}+\frac{s(l+1)}{s(l)}+1$$

If λ is the limiting ratio of (3.3), then as $l\to\infty$ from (1.4), we get

$$\lambda^{15}-\lambda^{14}-\lambda^{13}-\lambda^{12}-\lambda^{11}-\lambda^{10}-\lambda^{9}-\lambda^{8}-\lambda^{7}-\lambda^{6}-\lambda^{5}-\lambda^{4}-$$

. (3.6)

Hence, the limiting ratio of (3.3) is the positive real root of (3.4) which is **1.99997**approximately. (3.5)

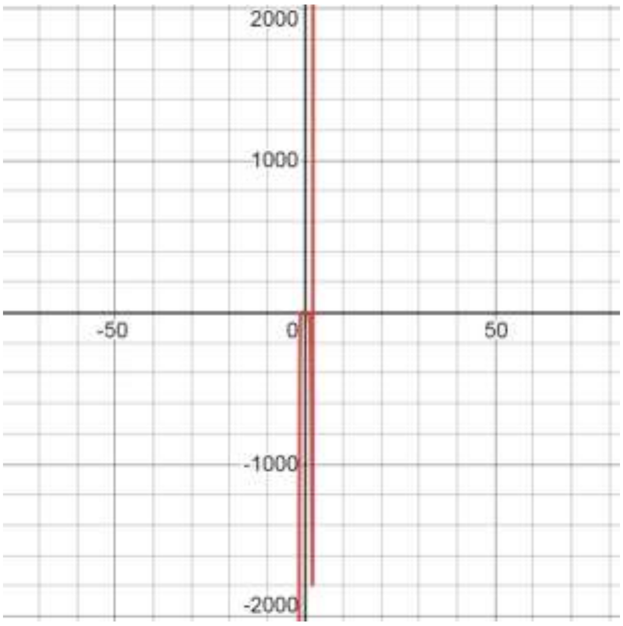

Figure4: Graph of y = $t^{15}-t^{14}-t^{13}-t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1$

When $m=18$

If $m=18$,then from (1.2), we get

$$s(l+18)=s(l+17)+s(l+16)+s(l+15)+.....+s(l+1)+s(l),l\geq 0,s(0)=s(1)=.....=s(17)=1$$

(3.7)

The characteristic equation of (3.7) is given by

$$t^{18}-t^{17}-t^{16}-t^{15}-t^{14}-t^{13}-t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1=0$$

(3.8)

By Iteration method, we see that the real root of the polynomial in is **2.000**approximately (3.9). Figure 5 verifies this fact.

From (3.7) we get

$$\frac{s(l+18)}{s(l)}=\frac{s(l+17)}{s(l)}+\frac{s(l+16)}{s(l)}+.......+\frac{s(l+2)}{s(l)}+\frac{s(l+1)}{s(l)}+1$$

If λ is the limiting ratio of (3.7), then as $l\to\infty$ from (1.4), we get

$$\lambda^{18}-\lambda^{17}-\lambda^{16}-\lambda^{15}-\lambda^{14}-\lambda^{13}-\lambda^{12}-\lambda^{11}-\lambda^{10}-\lambda^{9}-\lambda^{8}-\lambda^{7}-\lambda^{6}-\lambda^{5}-\lambda^{4}-\lambda^{3}-\lambda^{2}-\lambda-1=0$$

(4.0)

Hence, the limiting ratio of (3.7) is the positive real root of (3.8) which is **2.000**

approximately (3.9)

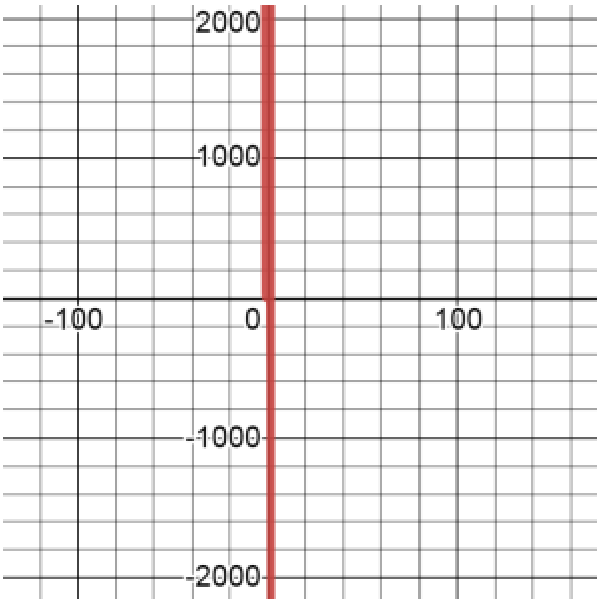

Figure5: Graph of y = $t^{18}-t^{17}-t^{16}-t^{15}-t^{14}-t^{13}-t^{12}-t^{11}-t^{10}-t^{9}-t^{8}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3}-t^{2}-t-1$

THEOREM

The limiting ratio of the generalized recurrence relation converges to 2

Proof: The generalized recurrence relation (as defined in (1.2)) is given by

$$s(l+m)=s(l+m-1)+s(l+m-2)+.....+s(l+1)+s(l), l\geq 0, m\geq 2$$
(4.1)

Where

$$s(0)=s(1)=s(2)=.....=s(m-2)=s(m-1)=1.$$

The characteristic equation of the generalized recurrence relation is given by

$$x^m-x^{m-1}-x^{m-2}-........-x^3-x^2-x-1=0 \qquad (4.2)$$

From, we have

$$\frac{s(l+18)}{s(l)}=\frac{s(l+17)}{s(l)}+\frac{s(l+16)}{s(l)}+.......+\frac{s(l+2)}{s(l)}+\frac{s(l+1)}{s(l)}+1$$

(4.3)s

Now using (1.4), as $n\to\infty$ (4.3) can be written as

$$\lambda^m-\lambda^{m-1}-\lambda^{m-2}-..........-\lambda^3-\lambda^2-\lambda-1=0 \quad (4.4)$$

We notice that (4.4) is the same equation as the characteristic equation of the generalized recurrence relation given by (4.2). Hence, the positive real root of the characteristic equation will be limiting ratio of the generalized recurrence relation given by (4.1).

If $f(\lambda)=\lambda^m-\lambda^{m-1}-\lambda^{m-2}-..........-\lambda^3-\lambda^2-\lambda-1.$

Then we find that $f(1)=-(m-1)<0$ since $m\geq 2$.

Similarly,

$$f(2)=2^m-2^{m-1}-2^{m-2}-.....-2^3-2^2-2-1=2^m-(2^{m-1})=1>0$$

.Hence, the positive real root λ of (4.4) must lie between 1 and 2 for all $m\geq 2$.

Since,

$$(\lambda-1)\left(\lambda^{m-1}+\lambda^{m-2}+...+\lambda^3+\lambda^2+\lambda+1\right)=\lambda^{m-1}$$

(4.5),

Using (4.4) in (4.5), we get $(\lambda-1)\lambda^m=1$.Simplifying this equation, we get $1+\lambda^{m+1}=2\lambda^m$ giving $\lambda+\frac{1}{\lambda^m}=2$ (4.6)

Since $\lambda>1, \frac{1}{\lambda^m}\to 0$ as $m\to\infty$ hence, from (4.6) we see that $\lambda\to 2$ as $m\to\infty$.

Thus, the limiting ratio λ of the generalized recurrence relation converges to 2.

This completes the proof.

CONCLUSION:

Generalizing the continued process of the Fibonacci sequence, we defined new continued process in (1.2). We proved this fact in this paper in the section 2.1 considering higher values of m as 11,12,13,15 and18 in sections 2.5, 2.9, 3.3and 3.7we obtained limiting ratio of each case. We also drew graph of the characteristic equation of all continued process in sections 2.1-3.7 to verify the obtained limiting ratio values. The roots were determined through Newton- Raphson method and graphs were constructed using Desmos graphing software – online software. In analyzing. the limiting ratio values obtained in section 3. We noticed that as we increase the value of m the limiting ratios were exactly nearer to1.9999 = 2. when, m=19 value of m the limiting ratios were exactly nearer to 2 in other side (2.0001). This fact was indeed proved to be true through theorem 1 of section 4.

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