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A STUDY ON STRONGLY REGULAR GRAPHS

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Abstract

In this paper, the graphs which were regular are discussed to be strongly regular graphs. Also certain theorems were proved based on strongly regular graphs with suitable examples.

Keywords: Cycle, Eigen values, Graph, Strongly regular Graph.

1. Introduction:

In Mathematics field, Graph theory is the learning of Graphs that means the tie between points and lines as vertices and edges. A Graph is a graphic description of a group of things where pair of things are merged by links. Graph Theory are also applied in Computer Science, Electrical Engineering, Physics and Chemistry.

Algebraic Graph Theory is a branch of Mathematics that involves to find the solutions for Algebraic Methods by using Graph Theory concepts. Linear Algebra, Group Theory and the study of Graph invariants are three main divisions of Algebraic Graph Theory. There is a bind between Graph Theory and Group Theory, which is shown by Arthur Cayley. He was the first to introduce the *Cayley Graphs* to finite groups.

We define Graph as a pair (V, E), where V is a non-empty set and E is the set of edges. A Graph is called Regular Graph if all the vertices having the same number of degrees. 0-regular graph is a empty or null graph. 1-regular graph is always a disconnected Graph. 3-regular graphs are also called Cubic Graph.

The main aim of this paper is to study on strongly regular graphs. Also we derived some theorems on Strongly Regular Graphs with suitable examples.

2. Preliminaries

Definition 2.1

A **Graph** X consists of a vertex set $V(X)$ and an edge set $E(X)$, where edge is an unordered pair of distinct vertices of X.

Definition 2.2

A graph in which degree of all the vertices are same is called as regular graph. If all the vertices in a graph are of degree k, then it is called ***k*-regular graph.**

Definition 2.3

An **undirected graph** is a graph without any direction of edges.

Example 2.1:

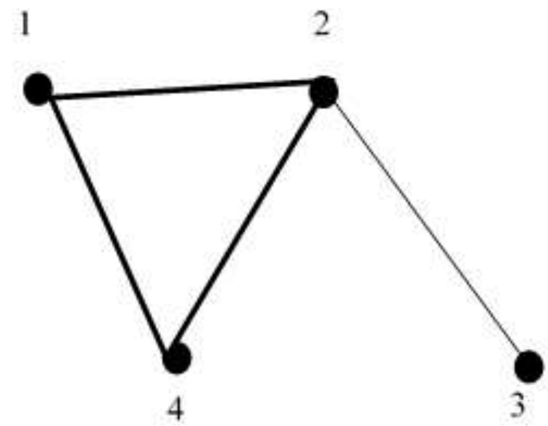

Figure 2.1 – Undirected Graph

In the Figure 2.1, the vertices are 1,2,3,4. (1,2), (1,4), (2,3), (2,4) are the edges of the graph. Since it is undirected graph the edges (1,2) and (2,1) are same. Similarly other edges are also considered in the same way.

Definition 2.4

The **eccentricity** $ecc(v)$ of a vertex v in X is the maximum distance from v to any other vertex $u \in X$.

Definition 2.5

The **diameter** of a graph X is denoted by $diam(X)$ and is defined by

$$diam(X) = max\,\{ecc(v), for\ all\ v \in X\}$$

Definition 2.6

Let $X = (V, E)$ be a graph with $V = (v_1, v_2, \dots, v_n)$, $E = (e_1, e_2, \dots, e_m)$ and without parallel edges. The **adjacency**

matrix of X is an $n\ x\ n$ symmetric binary $A = [a_{ij}]$ defined as integers such that

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in \text{E}, \\ 0, & \text{otherwise} \end{cases}$$

Example 2.2:

Consider the graph

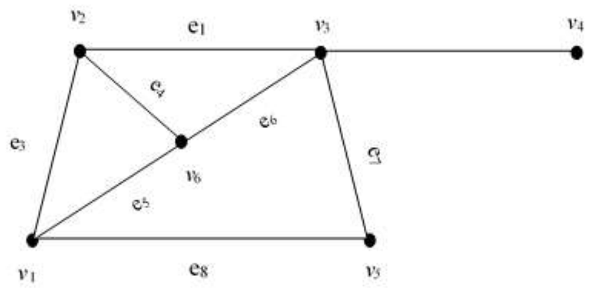

Figure 2.2 – Graph X

The adjacency matrix of X is given by

A=

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	1	1
v_2	1	0	1	0	0	1
v_3	0	1	0	1	1	1
v_4	0	0	1	0	0	0
v_5	1	0	1	0	0	0
v_6	1	1	1	0	0	0

Example 2.3:

The adjacency matrix of the Petersen graph of order 10 represented in Figure 3.1 is as follows:

B =

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_1	0	0	1	1	0	1	0	0	0	0
v_2	0	0	0	1	1	0	1	0	0	0
v_3	1	0	0	0	1	0	0	1	0	0
v_3	1	1	0	0	0	0	0	0	1	0
v_4	0	1	1	0	0	0	0	0	0	1
v_5	1	0	0	0	0	0	1	0	0	1
v_6 v_7	0	1	0	0	0	1	0	1	0	0
v_8	0	0	1	0	0	0	1	0	1	0
v_9	0	0	0	1	0	0	0	1	0	1
v_{10}	0	0	0	0	1	1	0	0	1	0

Definition 2.7

The graph X on n vertices is **strongly regular** with parameters (n, k, λ, μ) if

i) X is *k-regular*, such that every vertex in V has k neighbours.
ii) Each pair of adjacent vertices has exactly λ common neighbours.
iii) Each pair of non-adjacent vertices has exactly μ common neighbours.

Strongly regular graphs are denoted by $srg(n, k, \lambda, \mu)$.

Example 2.4:

The cycle graph with 5 vertices is represented in figure 3

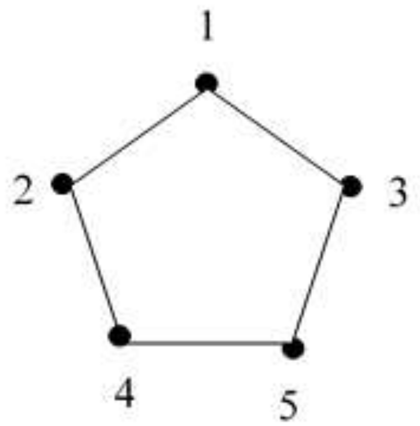

Figure 2.3 – Cycle C_5

Here, srg is (5, 2, 0, 1)

Definition 2.8

An **Eigen vector** of a $n \times n$ matrix A is a non-zero vector x such that $Ax = \lambda$ for some scalar λ. The scalar λ is the **Eigen value** of A.

3. Strongly Regular Graphs

Theorem 3.1

If $X = srg(n, k, \lambda, \mu)$ then $k(k - \lambda - 1) = (n - k - 1)\mu$

Proof:

Given X is a strongly regular graph with n vertices with k neighbours

Let u be a vertex in G

There are two ways to find the number of edges between the neighbours and non-neighbours of u.

Case: (i)

u has a set of k neighbours.

Let $N = k$ neighbours of u

Where $v \in N$ which is adjacent to u

And also v is adjacent to the non-neighbours of u.

There are $k - \lambda - 1$ non neighbours and there are k neighbours of u.

Therefore, $k(k - \lambda - 1)$ non neighbours of u.

Case: (ii)

Every $n - k - 1$ non neighbours of u is adjacent to μ.

Therefore, $\mu(n - k - 1)$ non neighbours of u.

Hence $k(k - \lambda - 1) = (n - k - 1)\mu$.

Example 3.1:

The Petersen graph with 10 vertices is represented in the following figure

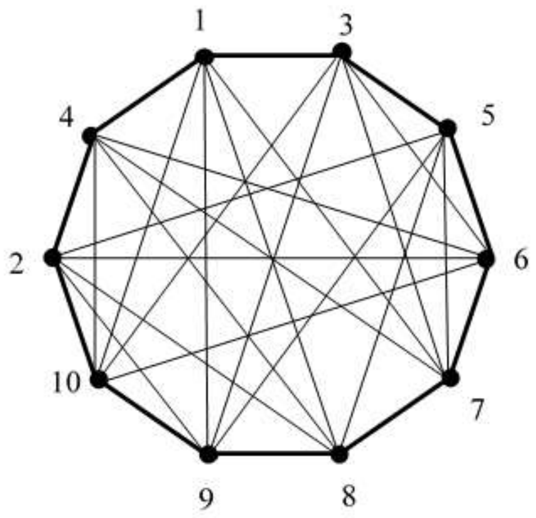

Figure 3.1 – Petersen Graph

Here, srg = (10, 3, 0,1)

Remark 3.1:

If X is strongly regular, then $\bar{X}$ is also strongly regular and

$$\bar{X} = srg\ (n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$$

Example 3.2:

Let us take Petersen graph, which is strongly regular graph and

srg = (10, 3, 0, 1)

The complement of Petersen graph is represented in Figure 3.2

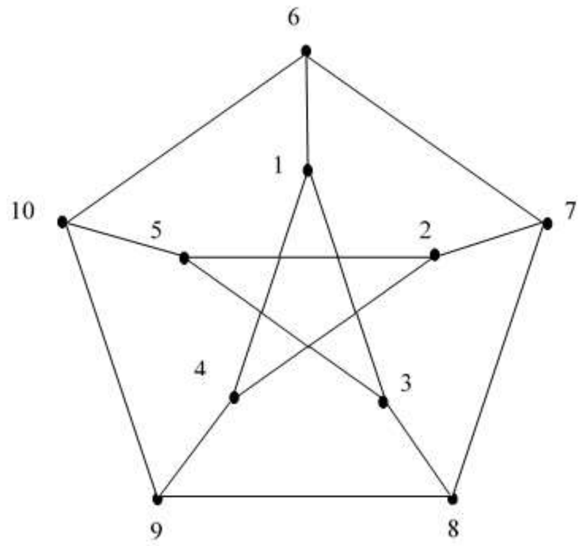

Figure 3.2 – Complement of Petersen Graph

Here, srg = (10, 6, 3, 4)

Note 3.1 :

Strongly regular graph is disconnected if and only if $\mu = 0$.

Theorem 3.2

Every connected strongly regular graph has diameter 2.

Proof:

Given X is a strongly regular graph with n vertices with k neighbours

Let u be a vertex in X

u is adjacent with some x_i where $x_i \in X$

And u is not adjacent with some y_i where $y_i \in X$

But x_i is adjacent with y_i

The eccentricity of u is 2 [By Definition 2.4]

Therefore, the diameter of any strongly regular graph is two.

Remark 3.2:

Let X be a strongly regular graph with parameters (n, k, λ, μ). Then the numbers

$f\ and\ g = \frac{1}{2}\left(n-1 \pm \frac{(n-1)(\mu-\lambda)-2k}{\sqrt{(\mu-\lambda)^2+4(k-\mu)}}\right)$ are non-negative integers.

Example 3.3:

Let us take the complement of the Petersen graph,

srg = (10, 6, 3, 4)

Then by the above remark 3.2 we get

$f = \frac{23}{6}$ and $g = \frac{31}{6}$

Here, $f\ and\ g$ are non-negative integers.

Theorem 3.3

Let X be a strongly regular graph with parameters (n, k, λ, μ) and Eigen values k, r and s. Then the complement of X (X_c) has eigenvalues (n-k-1), (-r-1) and (–s-1). Moreover the Eigen spaces of X_c and X are same.

Proof:

Let B be the adjacency matrix of X.

Then the adjacency matrix of X_c is $B^c = J - I - B$

We know that, k is an Eigen value of X and it's Eigen space is the space of constant vectors.

Let x be a constant vector.

We have $B^c = (J - I - B)\ x$

$= vx - x - kx$

$= (v - k - 1)\ x$

Therefore, $v - k - 1$ is an eigen value of X_c

And the eigen space is same as the eigen space of k.

Let y be an eigen vector of B corresponding to the eigen vector r.

We know that, y is orthogonal to j.

$B^c\ y = (J - I - B)\ y$

$= 0 - y - ry$

$= (-1 - r)y$

Therefore, $-r-1$ is an eigen value of X_c.

And it's eigen space is as same as the eigen space of r.

Similarly, $-1-s$ is an eigen value of X_c.

Hence the proof.

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