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SOLUTION OF LINEAR FREDHOLM INTEGRAL EQUATION SECOND KIND WITH TAYLOR EXPANSION USING MATLAB

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Abstract: In this paper, we approach to find the method to solve the Linear Fredholm Integral equation of second kind by Using Taylor Series Expansion in the Paper [1], the Kernel degenerate as a summation value of two function, where the equation of integral value is a=1, b=2 and this can be solved by using MATLAB software to get an appropriate value of the equation in easy way as compare to numerical methods.

Key words and phrases: Taylor series Expansion, Fredholm Integral Equation, MATLAB.

1. Introduction

Integral equation involving an unknown function which is under an integral sign. To solve the linear Fredholm integral equation in numerical methods it's more difficult, so that to get a solution of this equation is to use of the Taylor series Expansion for the function of two variables to get an appropriate solution for the Kernel K(x, t). From the paper [1], this explain that the fredholm integral equation of second kind with the integral value as a=0 and b=1.Now from that the paper [1], we are going to solve the integral value for a=1 and b=2.

Let us now consider the equation of linear Fredholm integral is

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{1.1}$$

Here we use the summation of multiplication function such as $g_i(t)$and $h_i(x)$ ie.

$$K(x,t) = \sum_{n=1}^{N} h_i(x)g_i(t) \tag{1.2}$$

From equation (1.2) use the degenerate Kernel idea. To solve the above equation, by the use of MAT lab software to deduced in an easy manner. This method is high accurate when compare the results with the other numerical methods solution.

Definition 1:

An integral equation is an equation in which the unknown function of t(x) to be determined appear under the integral sign. The equation is of the form,

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$$

Where K(x, t) is called the Kernel and a, b are the limit of the integration.

Definition 2:

The system of integral equation is said to be a *Linear* iff the linear

operator are performed in it upon the unknown function of t(x), then the equation is

$$L[y(x)] = f(x)$$

where $L[y(x)] = \int_a^b K(x,t)y(t)dt$. Then for any constants c_1 andc_2,

then$L[c_1y_1(x) + c_2y_2(x)] = c_1L[y_1(x)] + c_2L[y_2(x)]$ this the condition to satisfy the linear property.

Definition 3.

The Fredholm linear integral equation is

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt$$

where a and b are the limit of the integration, K(x, t) is the kernel and λ is a parameter.

The equation is called the linear integral equation.

Definition 4:

There are different kinds of Kernels

(1) Symmetric Kernel

(2) Seperable Kernel

(3) Iterated Kernel

(4) Resolvent Kernel

3. Objective of work:

The main idea to produce of a paper is an approximation solution by the Kernel with Taylor Series expansion for the function of two variables and making it to finding the solution of the Fredholm Integral equation.

4. Separable or Degenerate Kernel:

A Kernel K(x, t) is said to be a separable Kernel if it can be expressed as the sum of the finite number of terms S each of Kernel can be expressed as the product of two function such as a function of x only and function of t only i.e.,

$$K(x,t) = \sum_{n=1}^{N} g_i(x)h_i(t)$$

5. Solution of Fredholm Integral Equation of Second Kind with degenerate Kernel:

Consider a non-homogeneous Fredholm Integral equation of second kind.

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$$

∴ The Kernel is *degenerate*, then we take

$$K(x,t) = \sum_{n=1}^{N} h_i(x)g_i(t)$$

Assume that the function $h_i(x)$ is *Linearly Independent.*

Substitute equation (1.2) in equation (1.1) we get

$$y(x) = f(x) + \lambda \int_a^b [\textstyle\sum_{i=1}^{n} h_i(x)g_i(t)]y(t)dt \tag{5.1}$$

$$y(x) = f(x) + \lambda \textstyle\sum_{i=1}^{n} h_i(x) \int_a^b [g_i(t)y(t)]dt \tag{5.2}$$

Thus the above equation is reduces to,

$$y(x) = f(x) + \lambda \textstyle\sum_{i=1}^{n} c_i h_i(x) \tag{5.3}$$

where $c_i = \int_a^b g_i(t)y(t)dt$ (i=1,2,…....n) are constants to be determined in order to find the solution of (1.1) in the form of given by equation (5.3).

Now to proceed this to evaluate c_i as follows

$$y(t) = f(t) + \lambda \textstyle\sum_{i=1}^{n} c_i h_i(t) \tag{5.4}$$

substitute the values of y(t) in equation (5.2) we get,

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} h_i(x) \int_a^b g_i(t)[f(t) + \lambda \sum_{i=1}^{n} c_i h_i(t)]dt \quad (5.5)$$

Equating the equation (5.3) and equation (5.5)

$$f(x) + \lambda \sum_{i=1}^{n}[c_i h_i(x)] = f(x) + \lambda \sum_{i=1}^{n} h_i(x) \int_a^b g_i(t)[f(t) + \lambda \sum_{i=1}^{n} c_i h_i(t)]dt$$

Or $\sum_{i=1}^{n}[c_i h_i(x)] = \sum_{i=1}^{n} h_i(x) \int_a^b g_i(t)[f(t) + \lambda \sum_{i=1}^{n} c_i h_i(t)]dt$ (5.6)

Let $\beta_i = \int_a^b g_i(t) f(t) dt$ and $\alpha_{ij} = \int_a^b g_i(t) h_j(t) dt$ where α_{ij} and β_i are known as *constant*.

Then equation (5.6) is simplify as,

$$\sum_{i=1}^{n}[c_i h_i(x)] = \sum_{i=1}^{n} h_i(x) \left[\beta_i + \lambda \sum_{i=1}^{n} \alpha_{ij} c_j\right]$$

Or

$$\sum_{i=1}^{n} h_i(x) \left[c_i - \beta_i - \lambda \sum_{i=1}^{n} \alpha_{ij} c_j\right] = 0$$

By our assumption hi(x) are linearly independent,

$$\therefore c_i - \beta_i - \lambda \sum_{i=1}^{n} \alpha_{ij} c_j = 0, i = 1,2, \ldots .n; j = 1,2, \ldots .n.$$

$$c_i - \lambda \sum_{i=1}^{n} \alpha_{ij} c_j = \beta_i, i = 1,2, \ldots .n; j = 1,2, \ldots .n. \quad (5.7)$$

Thus we obtain a system of linear equations. To determine the value of$c_1, c_2, \ldots \ldots \ldots c_n$.

From the equation (5.7), put $i = 1$

$$c_1 = \lambda[\alpha_{11}c_1 + \alpha_{12}c_2 + \alpha_{13}c_3 + \cdots \ldots \ldots \ldots + \alpha_{1n}c_n]$$

put $i = 2$

$$c_2 = \lambda[\alpha_{21}c_1 + \alpha_{22}c_2 + \alpha_{23}c_3 + \cdots \ldots \ldots \ldots + \alpha_{2n}c_n]$$

and so on, In general put $i = n$

$$c_n = \lambda[\alpha_{n1}c_1 + \alpha_{n2}c_2 + \alpha_{n3}c_3 + \cdots \ldots \ldots \ldots + \alpha_{nn}c_n]$$

Thus the *determinate* of $D(\lambda)$ of system

$$D(\lambda) = \begin{vmatrix} (1-\lambda\alpha_{11}) & -\lambda\alpha_{12} & \ldots\ldots\ldots\ldots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & (1-\lambda\alpha_{22}) & \ldots\ldots\ldots\ldots & -\lambda\alpha_{2n} \\ \vdots & \vdots & \ldots\ldots\ldots\ldots & \vdots \\ \vdots & \vdots & \ldots\ldots\ldots\ldots & \vdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \ldots\ldots\ldots\ldots & (1-\lambda\alpha_{nn}) \end{vmatrix} \quad (5.8)$$

Which is a polynomial in λ of degree at most (n), $D(\lambda)$ is not *identically zero.*

Example 1: Find the analytical solution of the following integral equation

$$y(x) = 1 + \int_1^2 [1 + 4xt] y(t) dt$$

Solution. The given Integral equation is,

$$y(x) = 1 + \int_1^2 [1 + 4xt] y(t) dt$$

From the given integral equation we can choose the Kernel value as $K(x,t) = (1 + 4xt)$.

We know that the Kernel $K(x,t)$ is *Seperable function*. Thus$h_1(x) = 1, h_2(x) = 4x$

$g_1(t) = 1, g_2(t) = t\ f(x) = 1, \lambda = 1.$ From equation (5.4) we get,

$y(x) = 1 + [c_1 + 4xc_2]$

$$\begin{bmatrix} (1-\lambda\alpha_{11}) & -\lambda\alpha_{12} \\ -\lambda\alpha_{21} & (1-\lambda\alpha_{22}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

thus the value of λ is 1 then,

$$\begin{bmatrix} (1-\alpha_{11}) & -\alpha_{12} \\ -\alpha_{21} & (1-\alpha_{22}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

To find the valued of c_1 and c_2 we first find the α_{ij} ,$i \neq j$ and β_j , $(i = 1,21; j = 1,2)$,we get

$\alpha_{11} = \int_1^2 dx = 1, \alpha_{12} = \int_1^2 4x dx = 6, \alpha_{21} = \int_1^2 x dx = \frac{3}{2}, \alpha_{22} = \int_1^2 4x^2 dx = \frac{28}{3},$

$\beta_1 = \int_1^2 dx = 1, \beta_2 = \int_1^2 x dx = \frac{3}{2}$,Then substitute these values in the above matrix and we get

$$\begin{bmatrix} 0 & -6 \\ -\frac{3}{2} & -\frac{28}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$$

and the values of constants are $c_2 = -\frac{1}{6}, c_2 = -\frac{2}{27}$.

Thus the solution is $y(x) = 1 + [-\frac{2}{27} + (-\frac{2x}{3})]$.

6. Taylor series Expansion:

Let f(x, y) is a continuous function of two variable x and y, then the Taylor series Expansion of function f at the neighbourhood of any real number "a" with respect to y is

$taylor(f, y, a) = \sum_{n=0}^{\infty} \frac{(y-a)^n}{n!} \frac{\partial^n}{\partial y^n} f(x, y = a)$ And

$taylor(f, y, a) = \sum_{n=0}^{m-1} \frac{(y-a)^n}{n!} \frac{\partial^n}{\partial y^n} f(x, y = a)$

this mean that m^{th} term of Taylor expansion to the function at the neighbourhood

a With the respect to the variable y.

Example 2: Find the six terms of a Taylor series expansion of the function $f(x, y) = e^{xy}$ at

$a = 4.$

Solution: The given function is $f(x, y) = e^{xy}$ .The Taylor Expansion is

$taylor(f, y, a) = \sum_{n=0}^{m-1} \frac{(y-a)^n}{n!} \frac{\partial^n}{\partial y^n} f(x, y = a)$. To find the *four real number* with *six terms* where,

$$taylor(f,y,4,6) = e^{4x} + \frac{(y-4)}{1!}x\, e^{4x} + \frac{(y-4)^2}{2!}x^2\, e^{4x} + \frac{(y-4)^3}{2!}x^3 + \frac{(y-4)^4}{4!}x^4\, e^{4x} + \frac{(y-4)^5}{2!}x^5\, e^{4x}$$

Example 3: Compare the values of function $f(x,y) = e^{xy}$ at the point of (2, 3) with its Taylor expansion of 4 terms.

Solution: Consider the function $f(x,y) = e^{xy}$ and $f(2,3) = e^{2\times 3}$

$$f(2,3) = 403.42 \tag{6.1}$$

Now by find those values by using the Taylor expansion is,

$$taylor(f,x,2,4) = e^{2y} + (x-2)ye^{2y} + \tfrac{1}{2}(x-2)^2y^2e^{2y} + \tfrac{1}{6}(x-2)^3y^3e^{2y}. \tag{6.2}$$

$$taylor(f,x,2,4) = 403.42$$

from the above results show that the numerical value and by using the Taylor expansion are equal.

Remark 1: The Taylor series must be evaluated at a point or close to the point that the value we want for the function at that point as shown in example(3)

7. Main work:

For any *continuous function* $K(x,t)$ of two variables it can be approximated by *Taylor expansion*,

$$\therefore K(x,t) = \sum_{i=1}^{n} h_i(x)g_i(t)$$

Example 4: If $f(x,t) = e^{xt}$ then find the Taylor expansion with respect to variable t at $a = 0$ with 4 terms.

Solution: Consider the function $(x,t) = e^{xt}$.

The *Taylor Expansion* is

$$taylor(f,t,0,4) = 1 + tx + \frac{1}{2}t^2x^2 + \frac{1}{6}t^3x^3$$

From this $h_1(x) = 1,\ h_2(x) = x,\ h_3(x) = \frac{x^2}{2},\ \ h_4(x) = \frac{x^3}{6}$ and $g_1(t) = 1, g_2(t) = t,$ $g_2(t) = t,\ g_3(t) = t^2, g_4(t) = t^3.$

8. Algorithm:

Step-1: Start the program

Step-2: Declare the variables x.

Step-3: Calculate the α_{ij} $, i \neq j, (i = 1,2, \dots 5;\ j = 1,2, \dots 5)$values by using

$$\alpha_{ij} = \int_1^2 g_i(x)h_j(x)\,dx$$

Step-4: From the values of α_{ij} we can calculate the [A] value.

Step-5: Calculate the β_i $(\mathrm{i} = 1,2, \ldots 5)$ values by using $\beta_i = \int_1^2 g_i(x)\, f(x)\, dx$

Step-6: Calculate the Constants value $[c_i] = [A_i]^{-1} \times [\beta_i]^T$.

Step-7: Then the Solution of Y (x) is also calculated.

Step-8: Stop the Program.

9. Program:

Alpha values

>> syms x

>>α_{11} =int (sin (x), 1, 2)

α_{11} =0.9564

>>α_{12} =int (cos(x), 1, 2)

α_{12} = 0.0678

>>α_{13} =int ((- 0.5)* sin(x), 1, 2)

α_{13} = -0.4782

>>α_{14} =int ((- 0.1667)* cos(x), 1, 2)

α_{14}= -0.0113

α_{15}=int ((0. 0 4 1 7) * sin(x), 1, 2)

α_{15} =0.0399

>>α_{21} =int ((x)* sin (x), 1, 2)

α_{21}=1.4404

>>α_{22}=int ((x)* cos (x), 1, 2)

α_{22} = 0.0207

>> α_{23}=int ((x)*((-0.5)* sin (x)), 1, 2)

α_{23}= -0.7202

>> α_{24}=int ((x)*((-0.1667)* cos (x)), 1, 2)

α_{24} = -0.0034

>> α_{25}=int ((x) * ((0. 0 4 1 7) * sin (x)), 1, 2)

α_{25}=0.0601

>>α_{31}=int ((x ˆ2)*(sin (x)), 1, 2)

α_{31} = 2.2462

>> α_{32} =int ((x ˆ2)*(cos (x)), 1, 2)

α_{32} = -0.0851

>> α_{33}=i n t ((x ˆ2)*((-0.5)* sin (x)), 1, 2)

α_{33} = -1.1231

>> α_{34}=int ((x ˆ2)*((-0.1667)* cos (x)), 1, 2)

α_{34}= 0.0142

>> α_{35}=int ((x ˆ 2) * ((0 . 0 4 1 7) * sin (x)) , 1 , 2)

α_{35}= 0.0937

>> α_{41}=int ((x ˆ3)*(sin (x)), 1, 2)

α_{41}= 3.6141

>> α_{42}=int ((x ˆ3)*(cos (x)), 1, 2)

α_{42}= -0.3058

>> α_{43}=int ((x ˆ3)*((-0.5)* sin (x)), 1, 2)

α_{43}= -1.8070

>>α_{44}=int ((x ˆ3)*((-0.1667)* cos (x)), 1, 2)

α_{44}= 0.0510

>>α_{45}=int ((x ˆ 3) * ((0. 0 4 1 7) * sin (x)), 1, 2)

α_{45}= 0.1507

>> α_{51} =int ((x ˆ4)*(sin (x)), 1, 2)

α_{51} = 5.9754

>> α_{52} =int ((x ˆ4)*(cos (x)), 1, 2)

α_{52} = -0.7491

>>α_{53} =int ((x ˆ4)*((-0.5)* sin (x)), 1, 2)

α_{53} = -2.9877

>>α_{54} =int ((xˆ4)*((-0.1667)* cos(x)), 1, 2)

α_{54} = 0.1249

>>α_{55}=int ((x ˆ 4) * ((0 . 0 4 1 7) * s i n (x)) , 1 , 2)

α_{55}= 0.2492

Beta values

>>β_1 =int (1, 1, 2)

β_1 = 1

>>β_2 =int (x, 1, 2)

β_2 = 1.5000

>> β_3 =int ((x ˆ 2), 1, 2)

β_3 = 2.3333

>>β_4 =int ((x ˆ 3), 1, 2)

β_4 = 3.7500

>> β_5 =int ((x ˆ 4), 1, 2)

β_5= 6.2000

Constants value

A =0.0436   -0.0678   0.4782   0.0113   -0.0399

-1.4404	0.9793	0.7202	0.0034	-0.0601
-2.2462	0.0851	2.1231	-0.0142	-0.0937
-3.6141	0.3058	1.8070	-0.9490	-0.1507
-5.9754	0.7491	2.9877	-0.1249	0.7508

```
>> c= inv (A)* transpose (B)
c= 1.5195
   2.0528
   2.8646
  -4.5974
   6.1393
```

Approximation Solution

```
>> Y= 1+( ( (1.5195)* sin ( x )+( (2.0528)* cos ( x ) )+( (2.8646)*( -0.5)* sin ( x))+
((-4.5974)*(-0.1667)* cos (x))+( ( 6 . 1 3 9 3 ) * ( 0 . 0 4 1 7 ) * sin ( x ) ) ) )

Y =1+0.3432 sin (x) +2.8192 cos (x)
```

Example 5: Solve the Integral equation $y(x) = 1 + \int_1^2 \sin(x+s)\, ds$ with using MATLAB Software.

Solution: Consider the given Integral equation, $y(x) = 1 + \int_1^2 \sin(x+s)\, ds$.

Expand the $\sin(x+s)$ by *Taylor Series Expansion*.

$$taylor(\sin(x+s), s, 5) = \sin(x) + s\cos(x) - \frac{s^2}{2}\sin(x) - \frac{s^3}{6}\cos(x) + \frac{s^4}{24}\sin(x)$$

Which implies that $h_1(x) = \sin(x)$,$h_2(x) = cos(x)$ +$h_3(x) = -\frac{1}{2}\sin(x)$, $h_4(x) = -\frac{1}{6}\cos(x)$

$g_1(s) = 1$,$g_2(s) = s$,$g_3(s) = s^2$,$g_4(s) = s^3$,$g_5(s) = s^4$, from these values we are going find the $\alpha_{ij} = \int_1^2 g_i(x)h_j(x)\, dx$ for $i = 1,2,\ldots 5;\ j = 1,2,\ldots 5$ and $\beta_i = \int_1^2 g_i(x)\, f(x)\, dx$ for $i = 1,2,\ldots 5$ And constants value by using MATLAB.

In general,

$$\begin{bmatrix} (1-\lambda\alpha_{11}) & -\lambda\alpha_{12} & -\lambda\alpha_{13} & -\lambda\alpha_{14} & -\lambda\alpha_{15} \\ -\lambda\alpha_{21} & (1-\lambda\alpha_{22}) & -\lambda\alpha_{23} & -\lambda\alpha_{24} & -\lambda\alpha_{25} \\ -\lambda\alpha_{31} & -\lambda\alpha_{32} & (1-\lambda\alpha_{33}) & -\lambda\alpha_{34} & -\lambda\alpha_{35} \\ -\lambda\alpha_{41} & -\lambda\alpha_{42} & -\lambda\alpha_{43} & (1-\lambda\alpha_{44}) & -\lambda\alpha_{45} \\ -\lambda\alpha_{51} & -\lambda\alpha_{52} & -\lambda\alpha_{53} & -\lambda\alpha_{54} & (1-\lambda\alpha_{55}) \end{bmatrix}$$

from the matrix we have to first find the $\alpha_{ij} = \int_1^2 g_i(x)h_j(x)\, dx$, $(i = 1,2,\ldots 5;\ j = 1,2,\ldots 5)$ where $i \neq j$.

$$\alpha_{ij} = \begin{bmatrix} 0.9564 & 0.0678 & -0.4782 & -0.0113 & 0.0399 \\ 1.4404 & 0.0207 & -0.7202 & -0.0034 & 0.0601 \\ 2.2462 & -0.0851 & -1.1231 & 0.0142 & 0.0937 \\ 3.6141 & -0.3058 & -1.8070 & 0.0510 & 0.1507 \\ 5.9754 & -0.7491 & -2.9877 & 0.1249 & 0.2492 \end{bmatrix}$$

The values of A_i is

$$A_i = \begin{bmatrix} 0.0436 & -0.0678 & 0.4782 & 0.0113 & -0.0399 \\ -1.4404 & 0.9793 & 0.7202 & 0.0034 & -0.0601 \\ -2.2462 & 0.0851 & 2.1231 & -0.0142 & -0.0937 \\ -3.6141 & 0.3058 & 1.8070 & -0.9490 & -0.1507 \\ -5.9754 & 0.7491 & 2.9877 & -0.1249 & -0.7508 \end{bmatrix}$$

The values of β_i is

$$\beta_i = [1.0000 \quad 1.5000 \quad 2.3333 \quad 3.7500 \quad 6.2000]$$

Thus the constants values is found by $[c_i] = [A_i]^{-1} \times [\beta_i]^T$

$$c_{i=}[1.5195 \quad 2.0528 \quad 2.8646 \quad -4.5974 \quad 6.1393]$$

And the Approximation solution is

$$\boldsymbol{y(x) = 1 + 2.8192\cos(x) + 0.3432\ sin(x)}$$

10. Conclusion:

Solution of the Fredholm Integral of Second Kind by using Taylor Series Expansion With the use of MATLAB Software it gives an accurate value. We approached the integral equation with its integration limits from a = 1 and b = 2.It gives us the more evident results in this manner compare to another methods. From the paper [1], we changed the integration limits and get the approximation value by the method.

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