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SOLVING LINEAR NON HOMOGENEOUS DIFFERENTIAL EQUATIONS BY APPLYING MATRIX DIFFERENTIAL OPERATOR

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ABSTRACT

In this paper, the method of undetermined coefficients for the MDO (Matrix Differential Operator) was established. By using this operator we find the general solution, particular solution and the initial value condition. To check the matrix differential operator in a linear non homogeneous equation gives the detailed solution.

Key Words: Linear Non Homogeneous Differential Equations, Matrix Differential Operator.

INTRODUCTION

In Mathematics, Ordinary Differential equations (ODEs) have big importance and other field of science. A differential equation is an equation that relates one or more functions and their derivatives. Linear differential equations appear many fields such as science and engineering like network circuit analysis and mechanics etc. The linear non homogeneous differential equations are generally solved by many methods like Laplace transform and Fourier transform [1], [2], [3]. This paper uses the method of undermined coefficient, to take their formula for solving the linear non homogeneous differential equations in matrix differential operator method. Let us consider a differential equation with characteristic equation,

$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x' + a_0 = f(y), a_n \neq 0$ As

$(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)x = f(y)$

and then finding particular integral using (MDO) method. We find the particular integral with right hand side of the differential equation, as $y = x^k e^{ax}(U_m(x)\sin\beta x + V_m(x)\cos\beta x)$ [4]. we introduce the differential operator notation. It is sometimes convenient to adopt the notation $Dy, D^2y, D^3y, \dots, D^ny$ to denote $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$, the symbols $Dy, D^2y, \dots$ are called differential operators and have properties analogous to those of algebraic quantities [6].

After finding the particular integral in MDO method. We substitute these values into initial value condition and finding the values of A, B, C, D then the initial conditions in the general solution. Then the general equation becomes into the linear non homogeneous differential equation. In this paper, it has been applied a new technique called Matrix Differential Operator (MDO) formula to solve these equations.

MATRIX DIFFERENTIAL OPERATOR

If $S = v_1, v_2, \dots, v_n$ is a set of vectors in a vector space V, then the set of all linear combination of $v_1, v_2, \dots, v_n$ is called the span of $v_1, v_2, \dots, v_n$ and is denoted by span $(v_1, v_2, \dots, v_n)$ or span(S).Let G be a vector space of all differentiable function. Consider the subspace $V \subset G$ given by $V = span(f_1(x), f_2(x), \dots, f_n(x))$ where we assume that the function $f_1(x), f_2(x), \dots, f_n(x)$ are linearly independent. Since the set $B = \{f_1(x), f_2(x), \dots, f_n(x)\}$ is linearly independent, it is a basis for V. The functions $f_i(x), i = 1,2,\dots,n$ expressed in basis B using base vector coordinates are usually written.

$$[f_1(x)]_B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, [f_2(x)]_B = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, [f_n(x)]_B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

The vector $[f_i(x)]_B$ has in the ith row 1 and 0 otherwise. Further, assume that the differential operator D maps V into itself let $D,(f_1(x)) = \sum_{j=1}^{n} c_{ij} f_j(x), i = 1,2,\dots,n,$ where

$c_{ij} \in \mathbb{R}, i,j = 1,2,\dots,n$ are constants. Then

$$[D(f_1(x))]_B = \begin{pmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{in} \end{pmatrix}, i,j = 1,2,\dots,n \text{ and ([9])}$$

$$[D]_B [[D(f_1(x))]_B \vdots [D(f_1(x))]_B \vdots \cdots \vdots\ [D(f_1(x))]_B = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

If $f(x) = \sum_{i=1}^{n} \alpha_i f_i(x), \alpha_i \in \mathbb{R}, i = 1,2,\dots,n, f(x) \in V$ then $[f(x)]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

We express what is the derivative of the function f(x)

$Df(x) = \sum_{i=1}^{n} \alpha_i Df_i(x) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} c_{ij} f_j(x) = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_i c_{ij} f_j(x)$ respectively

$$[D(f_1(x))]_B == \begin{pmatrix} \sum_{i=1}^{n} c_{i1}\, \alpha_1 \\ \sum_{i=1}^{n} c_{i2}\, \alpha_2 \\ \vdots \\ \sum_{i=1}^{n} c_{in}\, \alpha_n \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Let us next simply denote $[D]_B$ as D_B.The matrix D_B we will called a matrix differential operator corresponding to a vector space V with the considered basis B.

Denote

$[D(f_1(x))]_B = f'_B = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$ and $[f(x)]_B = f_B$ then $f'_B = D_B f_B$ and [4].

Example 1. Use matrix differential operator to solve

$(D^2 + k^2)^2 x = \cos kt \ if\ x, Dx, D^2 x\ are\ zero\ at\ t = 0$

Solution:

$Given, (D^2 + k^2)^2 x = \cos kt$

$(or) x'''' + 2k^2 x'' + k^4 x = \cos kt$

The general form of differential equations as,

$a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x^1 + a_n x = f(t)$

Then the characteristic equations as,

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

$$\lambda^4 + 2k^2\lambda^2 + k^4 = 0$$

$$(i.e)(\lambda^2 + k^2)^2 = 0$$

Find roots $\lambda_1 \ldots \lambda_n$,where k-multiplicity of the root,

$$\lambda^4 + 2k^2\lambda^2 + k^4 = 0$$

$$\lambda^2(\lambda^2 + k^2) + k^2(\lambda^2 + k^2) = 0$$

Therefore , $\lambda = \pm ik \;, \lambda = \pm ik$

And therefore the roots are imaginary $(\alpha + i\beta)$, $since\ \alpha = 0,$

$$= e^{\alpha t}(m_1 cos\beta t + m_2 sin\beta t)(m_3 + m_4 t)$$

$$= At \sin kt + Bt \cos kt + C \sin kt + D \cos kt$$

$$where\ A = m_2 m_4\ , B = m_1 m_4\ , C = m_2 m_3\ , D = m_1 m_3$$

General solution of the form,

$$x = \sum P_{k-1}(t) e^{\alpha t} \sin \beta t + \sum Q_{k-1}(t) \alpha t \cos kt$$

$$where, \lambda = \alpha \pm i\beta\ and\ P_{k-1}(t), Q_{k-1}(t) \rightarrow C_1 + \cdots + C_k t^{k-1}$$

$$C.F(x_1) = At \sin kt + Bt \cos kt + C \sin kt + D \cos kt$$

Using matrix differential operator,

$$e^{\alpha t}(P_m(t) \cos \beta t + Q_m(t) \sin \beta t)$$

where $f(t) = \cos kt,$Then P.I of the form,

$$y = x^k e^{\alpha x}(U_m(t) \sin \beta t + V_m(t) \cos \beta t)$$

And then the particular solution of vector space,

$$V = span(t^2 \sin kt\ , t^2 \cos kt, t \sin kt, t \cos kt, \sin kt, \cos kt)$$

$$And, B = \{t^2 \sin kt, t^2 \cos kt, t \sin kt, t \cos kt, \sin kt, \cos kt\}$$

The above equation can be differentiate with respect to "t" as,

$$D_B = A(2t \sin kt + t^2 k \cos kt) + B(2t \cos kt - t^2 k \sin kt) \\ + C(\sin kt + \text{tk} \cos kt) + D(\cos kt - tk \sin kt) + E\,(k \cos kt) + F(-k \sin kt)$$

$$= (-Bk)t^2 \sin kt + (Ak)t^2 \cos kt + (2A - Dk)t \sin kt + (2B + Ck)t \cos kt \\ + (C - Fk) \sin kt + (D + Ek) \cos kt$$

The matrix differential operator,

$$D_B = \begin{pmatrix} 0 & -k & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -k & 0 & 0 \\ 0 & 2 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -k \\ 0 & 0 & 0 & 1 & k & 0 \end{pmatrix}$$

Then, $\left(D_B{}^4 + 2k^2 D_B{}^2 + k^4 I_6\right)X_B = \cos kt$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -8k^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8k^2 & 0 & 0 & 0 & 0 \end{pmatrix} X_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -8k^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8k^2 & 0 & 0 & 0 & 0 \end{pmatrix}^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{-1}{8k^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{8k^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-1}{8k^2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$X_B = \frac{-t^2 \cos kt}{8k^2}$$

$$x = x_1 + X_B = At \sin kt + Bt \cos kt - \frac{t^2 \cos kt}{8k^2} + C \sin kt + D \cos kt$$

Find the value of A, B, C, D then the initial conditions in the general solution,

$$x = x_1 + X_B = At \sin kt + Bt \cos kt - \frac{t^2 \cos kt}{8k^2} + C \sin kt + D \cos kt \longrightarrow *$$

$$at, t = 0, x = x' = x'' = x''' = 0$$

$$x = 0 = 0 + C \sin 0 + 0 + D \cos 0$$

$$D = 0 \qquad \rightarrow (1)$$

$$x' = 0 = A \sin 0 + 0 + C\ k \cos 0 - 0 + 0 + B \cos 0 - 0 - Dk \sin 0$$

$$Ck + B = 0 \qquad \rightarrow (2)$$

$$x'' = 0 = 2Ak \cos 0 - 0 - Ck^2 \sin 0 + 0 - \frac{\cos 0}{4k^2} + 0 - 2Bk \sin 0 - Dk^2 \cos 0$$

$$2Ak - \frac{1}{4k^2} - Dk^2 \quad \rightarrow (3)$$

$$x''' = 0 = -3Ak^2 \sin 0 - 0 - Ck^3 \cos 0 + 0 + \frac{3 \sin 0}{4k} - 0 - 3Bk^2 \cos 0 + Dk^3 \sin 0 \rightarrow (4)$$

Solving the above equation (1), (2), (3), (4) we get,

$$A = \frac{1}{8k^3}, B = C = D = 0, in\ equation\ (*) we\ get,$$

$$x = \frac{t \sin kt}{8k^3} - \frac{t^2 \cos kt}{8k^2}$$

2. Use matrix differential operator to solve

$(D^4 - 1)x = 1, if\ x = Dx = D^2x = D^3x = 0$

Solution:

$Given, (D^4 - 1)x = 1$

$(or) x'''' - x = 1$

Then the characteristic equations as,

$\lambda^4 - 1 = 0 \Longrightarrow (\lambda - 1)(\lambda + 1)(\lambda^2 + 1) = 0$

The roots are $\lambda = -i, i, 1, -1$

General solution, $C.F(x_1) = A \sin t + B \cos\ t + C\ e^t + D\ e^{-t}$

Using matrix differential operator

Where f (t) =1, $\alpha + i\beta = 0,$

$x_i = t^s e^{\alpha t}(R_m(t) \cos \beta t + T_m\ (t) \sin \beta t)$

$V = span(\sin(t), \cos(t))$

$B = \{\sin(t)\ , \cos(t)\}$

The above equation can be differentiate with respect to "t" as, t=0

$D_B = (\cos(0))A + (-\sin(0))B \Longrightarrow (-B) \sin(0) + (A) cos(0) = (-B)0 + (A)1$

$D_B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$(D_B{}^4 - I_2)X_B = 1$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} X_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$X_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$X_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$X_B = -1$

$x = x_1 + X_B = A \sin t + B \cos t + ce^t + De^{-t} - 1$

Find the value of A, B, C, D then the initial conditions in the general solution,

$x = A \sin t + B \cos t + C\ e^t + De^{-t} - 1, at\ t = 0, x = x' = x'' = x''' = 0 \rightarrow (*)$

$x = 0 = A\ sin\ 0 + B \cos 0 + Ce^0 + De^0 - 1$

$B + C + D - 1 = 0 \quad \rightarrow (1)$

$x' = 0 = A \cos 0 - B \sin 0 + Ce^0 - De^0$

$A + C - D = 0 \quad \rightarrow (2)$

$x'' = 0 = 0 - B \cos 0 + Ce^0 + De^0$

$-B + C + D = 0 \quad \rightarrow (3)$

$x''' = 0 = -A \cos 0 + B \sin 0 + Ce^0 + De^0$

$-A + C - D = 0 \quad \rightarrow (4)$

Solving the above equation (1), (2), (3), (4) we get,

$$A = 0, B = \frac{1}{2}, C = \frac{1}{4}, D = \frac{1}{4} \text{ in equation}(*), \text{we get}$$

$$x = \frac{\cos t}{2} + \frac{e^t}{4} + \frac{e^{-t}}{4} - 1 (or)$$

$$x = \frac{\cos t}{2} + \frac{\cos ht}{2} - 1$$

CONCLUSION

In this paper, we analyzed the linear non homogeneous differential equations in matrix differential operator. We need to calculate the general solution, particular solution and initial value condition. The linear non homogeneous differential equations by expressing a matrix differential operator are illustrated.

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