ISSN: 0974-5823 Vol. 7 No. 4 April, 2022

International Journal of Mechanical Engineering

FEKETE SZEGÖ PROBLEM FOR LOGARITHMIC COEFFICIENTS OF CERTAIN ANALYTIC FUNCTION

[1]K. DHANALAKSHMI and R.SANGEETHA ARASI[2]

PG & RESEARCH DEPARTMENT OF MATHEMATICS, THEIVANAI AMMAL COLLEGE FOR WOMEN (A), VILLUPURAM, TAMILNADU, INDIA

Abstract:

In this paper, we obtain the fekete szegö function from the subclasses a_2 ,a_3 of Logarithmic coefficients of analytic function.

Keywords: Analytic function, Univalent function, Logarithmic coefficients, Fekete Szegö.

1 Introduction and Preliminaries

Let $\mathcal{D}$ denote the class of functions f(z). A function $f \in \mathcal{D}$ has the taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

which is analytic in the unit disk $\mathbb{B}: z \in \mathbb{C}: |z| < 1$. Let S be the class of all function $f \in \mathcal{D}$ that are univalent (i.e.,one-to-one)in $\mathcal{D}$. For a general theory of univalent functions, For refer the classical books [3],[7] also

let P be the class of all analytic function p in $\mathbb{B}$, with the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \qquad (1.2)$$

such that $Re\{p(z)\} > 0$. for all $z \in \mathbb{B}$. A member of P is called a caratheodory function. it is known that $\|c_n\| \le 2\,, n \ge 1$ for a function $p \in \mathbb{P}$ (see [3]).

The Logarithmic coefficients of a function $f \in \mathcal{D}$ are defined in $\mathbb{B}$ by the following series expansion

$$log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n,$$

where γ_n are known as the Logarithmic coefficients. The coefficient γ_n have a great importance to play a central role in a theory of univalent function. For exact upper bounds for $|\gamma_n|$ have been established. Millin conjecture [10] that for $f \in \mathbb{S}\ and\ n \ge 2$

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k}\right) \le 0$$

where the equality holds if, and only if, rotation of the koebe function . For the Koebe function $k(z) = z/(1-z)^2 (z \in \mathbb{B})$, the coefficients γ_n, since the koebe function k(z) plays a role of extremal function in the class S. It is expected that $|\gamma_n| \leq -\frac{1}{n}$ and the Logarithmic coefficients is that finding the sharp bound of $|\gamma_n|$ for the class S [1],[2],[6] such that $|\gamma_1|$ and by differentiating (1.3) and equating coefficents, we obtain

$$|\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) = 0.635... \qquad (1.3)$$

the sharp bounds of γ_n , for $n \geq 3$ for the class S.

we have to taking the subclasses from the classess of function $S_\beta(\alpha), \mathcal{G}(\gamma), \mathrm{F}_0(\lambda)$ for obtaining Fekete szegö function

A function $f \in \mathcal{D}$ is called starlike if $f(\mathbb{B})$ is a starlike domain with respect to origin. Function satisfying Spacek condition [15] have been called spiral-like, and f is regular in $\mathbb{B}$ and The family $S_\beta(\alpha)$ of $\beta - spirallike$ function of order α is defined by

$$S_\beta(\alpha) = \left\{f \in \mathcal{D}: Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha cos\beta\right\}$$

where $0 \leq \alpha < 1$ and $\frac{-\Pi}{2} < \beta < \frac{\Pi}{2}$. such that $S_\beta(\alpha)$ is univalent in $\mathbb{B}$ (see [9])

A function $f \in \mathcal{D}$ is said to be locally univalent function at a point $z \in \mathbb{B}$. A family $\mathcal{G}(\gamma)$, $v > 0$ of function is defined by

$$\mathcal{G}(\gamma) = \left\{f \in \mathrm{U}: Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{v}{2}\right\}$$

the class $\mathcal{G}(\gamma) : \mathcal{G}(\infty)$ was first introduced by Ozaki [11]

Let $f \in \mathcal{D}$ be a locally univalent then by the Kalplan characterization it follows that f is close-to-convex in $\mathbb{B}$ and f is univalent in $\mathbb{B}$. The class $F(\lambda)$ has also been considered for the restriction $\frac{1}{2}$, denote by $\mathrm{F}_0(\lambda)$ [12],[13],[14].

The Fekete szegö inequalities introduced in 1933, preoccupied researchers regarding different classes of univalent functions [4],[8]. The Fekete szego problem is the problem of maximizing the absolute value of the functional in subclasses of normalized functions is called Fekete szego problem.

$$|a_3 - \mu a_2^2|$$

for various subclasses of univalent function. To know much more of history, we refer the readers to [8],[16]. The classical Fekete szegö functional is defined by

$$\Lambda_\mu = a_3 - \mu\mu a_2^2 (0 < \mu < 1)$$

The mathematicians who introduced the functional, M.Fekete and G.Szego [5], were able to bound the classical function in the class S by $1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right)$

The main purpose of this paper is to obtain Fekete Szegö function and its inequalities from the subclasses of logarithmic coefficient of certain analytical function.

2 Preliminaries

Lemma 1. *If* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots, (z \in \mathbb{B})$ *is a function with positive real part, then for any complex number* η*,*

$$|c_2 - \eta c_1^2| \leq 2\max\{1, |2\eta - 1|\}, \qquad (2.1)$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2},\ p(z) = \frac{1+z}{1-z}$$

3 Main Results

Theorem 1 *let* $-\Pi/2 < \beta < \Pi/2$ *and* $0 \leq \alpha < 1$. *For every* $f \in \mathbb{S}_\beta(\alpha)$ of the form (1.1) *then, we consider the subclasses* a_2 *and* a_3 *we have,*

$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}\left|e^{i(\beta-\alpha\beta)} cos\beta c_2 - \left(\frac{2\lambda e^{2i(\beta-\alpha\beta)} - (1-\alpha)^2 e^{2i\beta}}{2}\right) cos^2\beta c_1^2\right|$$

where

$$a_2 = (1-\alpha)e^{i\beta} cos\beta c_1,\ a_3 = \frac{(1-\alpha)^2 e^{2i\beta} cos^2\beta c_1^2 + (1-\alpha)e^{i\beta} cos\beta\, c_2}{2}$$

Proof. Let $f \in \mathbb{S}_\beta(\alpha)$ and we have their subclasses a_2 and a_3, we consider fekete szego functional, and

substitution of a_2 and a_3 in fekete szegö functional, we get

$$|a_3 - \lambda a_2^2| = \left|\frac{(1-\alpha)^2 e^{2i\beta} cos^2\beta\, c_1^2 + (1-\alpha)e^{i(\beta-\alpha\beta)} cos\beta\, c_2}{2} - \lambda e^{i(\beta-\alpha\beta)} cos\beta c_1\right| \qquad (3.1)$$

therefore to simplify the above equation, we get

$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}\left|e^{i(\beta-\alpha\beta)} cos\beta c_2 - \left(\frac{2\lambda e^{2i(\beta-\alpha\beta)} - (1-\alpha)^2 e^{2i\beta}}{2}\right) cos^2\beta c_1^2\right| \qquad (3.2)$$

This completes the proof

Theorem 2. *If $f \in \mathbb{G}(v)$ given by (1.1) then, we have the subclasses*

a_2 and a_3

$$|a_3 - \mu a_2^2| \leq -\frac{1}{6}\max\left\{v, \left|\frac{12\mu v^2 - 8v^2 - 8}{8}\right|\right\} \quad (3.3)$$

Proof. let $f \in \mathbb{G}(v)$, then there exist a caratheodory function p of the form

$v(p(z) - 1)f'(z) = -2zf''(Z)$

by using Taylor series for function f and p and equating the coefficient of z and z^2 in above equation, we obtain

$$a_2 = \frac{vc_1}{4}, a_3 = \frac{v^2c_1^2 - 2vc_2}{24} \quad (3.4)$$

by substituting a_2 and a_3 in the fekete szegö functional we get

$$|a_3 - \mu a_2^2| = \left|\frac{v^2c_1^2 - 2Vc_2}{24} - \mu\left(\frac{vc_1}{4}\right)^2\right| \quad (3.5)$$

then, Further Simplification gives

$$|a_3 - \mu a_2^2| = -\frac{1}{12}\left|vc_2 + \left(\frac{12\mu v^2 - 8v^2}{16}\right)c_1^2\right| \quad (3.6)$$

by using lemma , we get

$$|a_3 - \mu a_2^2| \leq -\frac{1}{6}\max\{v, |\frac{12\mu v^2 - 8v^2 - 8}{8}\}| \quad (3.7)$$

This completes the proof.

Theorem 3 *Let $f \in F_0(\lambda)$, for $\frac{1}{2} \leq \lambda \leq 1$, given by (1.1). Then*

$$a_2 = \frac{2\lambda+1}{4}c_1,\ a_3 = \frac{2\lambda+1}{24}(2c_2 + (2\lambda+1)c_1^2) \quad (3.8)$$

and for any complex number ξ

$$|a_3 - \xi a_2^2| \leq \frac{1}{6}\max\left\{2\lambda + 2, \left|\frac{\xi(6\lambda+3)^2 - (4\lambda+2)^2 - 2}{2}\right|\right\} \quad (3.9)$$

Proof. Let $f \in F_0(\lambda)$ be of the form(1.1), then

$$|a_3 - \xi a_2^2| = \left|\frac{(4\lambda+2)c_2+(2\lambda+1)^2c_1^2}{24} - \xi\left(\frac{2\lambda+1}{4}\right)^2 c_1^2\right| \tag{3.10}$$

rearranging and simple calculation we get

$$|a_3 - \xi a_2^2| = \frac{1}{12}\left|(2\lambda+2)c_2 - \frac{\xi(6\lambda+3)^2-(4\lambda+2)^2}{4}c_1^2\right| \tag{3.11}$$

and by using (1.5)and after some computations, we obtain

$$|a_3 - \xi a_2^2| \le \frac{1}{6}\max\left\{2\lambda+2, \left|\frac{\xi(6\lambda+3)^2-(4\lambda+2)^2-2}{2}\right|\right\} \tag{3.12}$$

Hence, completes the proof

References

[1] M. F. ALI and A. VASUDEVARAO, On logarithmic coefficients of some close-to-convex functions, Proc. Amer. Math. Soc. 146 (2018), 1136-1142.

[2] M. F. ALI and A. VASUDEVARAO, On logarithmic coefficients of some close-to-convex functions, Bull. Aust. Math. soc. 97(2) (2018), 253-264.

[3] P. L. DUREN, Univalent Functions, Springer-Verlag, New York, 1983.

[4] Dziok,J. A general solution of the Fekete-Szego problem.Bound Value problem. 2013,98,13.

[5] Fekete,M,Szego,G: Eine Bemerkung uber ungerade schichte Funktionen. J. Lond. math. Soc. 8,85-89(1933).

[6] D. GIRELA, Logarithmic coefficients of univalent functions , Ann. Acad. Sci. Fenn. Math. 25 (2000), 337-350

[7] A. W. GOODMAN, Univalent Functions, Vols. 1-2, Mariner Publishing Co., Tampa, FL, 1983.

[8] S. Kanas, An unified approach to Fekete-Szego problem, Appl. Math. Compt, **218**(2012), 8453-8461.

[9] R. J. LIBRA, Univalent $\alpha - spiral$ functions, Canad. J. Math. **19** (1967), 449-456.

[10] I. M. MILIN, Univalent Function and Orthonormal Systems, Izdat. "Nauka" , Moscow, 1971 (in Russian); English transl. American Mathematical Society, Providence (1977).

[11] S. OZAKI, On the theory of multivalent functions. II, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4** (1941), 45-87.

[12] J. A. PFALTZGRAFF, M. O. READE, and T. UMEZAWA, Sufficient condition for univalence, Ann. Fac. Sci. Univ. Nat. Zaire (Kinshasa) Sect. Math.-Phys. **2**(2) (1976), 211-218.111

[13] S. PONNUSAMY, S. K. SAHOO, and H. YANAGIHARA, Radius of convexity of partial sums of functions in the close-to-convex family, Nonlinear Anal. **95** (2014), 219-228.

[14] M. S. ROBERTSON, On the theory of univalent functions, Ann. Math. (*2*) **37**(2) (1936), 374-408.

[15] L. ŠPAČEK, Contribution à la thèorie des fonctions univalentes (in Czech), Časop pěst. Mat.-Fys. **62** (1933), 12-19.

[16] F. Yousef, R. A. Frasin, T. A Hawary, Fekete-Szego inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials, Filomat, 32(2018), 3229-3236.