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The Fekete-Szegö Inequality for new subclass of bi-univalent function

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Abstarct:

In this present paper,we consider new subclass of Bi-univalent function related with Legendre polynomials and the Fekete-Szegö Inequality for the new subclass is solved.

Keywords: Legendre polynomials, Bi-univalent functions, Subordination, Fekete-Szegö Inequality.

1.Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions f in an open disc $\mathcal{U} = \{z: z \in \mathbb{C}: |z| < 1\}$ normalized by the following Taylor-Maclaurin series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathcal{U}) \qquad (1.1)$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consist of univalent functions in $\mathcal{U}$. The important member of the class $\mathcal{S}$ is the Koebe function

$$K(z) = z(1-z)^{-2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} nz^n, \quad for z \in \mathcal{U}.$$

The Koebe function maps $\mathcal{U}$ in a one-to-one manner onto the domain D that consists of the entire complex plane except for a slit along the negative real axis from

$w = -\infty \quad to \quad w = -\frac{1}{4}.$

Let the function,

$$\phi(z) = \frac{1-z}{\sqrt{1-2z\cos\alpha+z^2}},$$

is in $\mathcal{P}$ for every real α [6] , where $\mathcal{P}$ is the Caratheodory class defined by

$$\mathcal{P} = \{p(z): \Re(p(z)) > 0, z \in \mathcal{U}\},$$

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots.$$

then,

$$\phi(z) = 1 + \sum_{n=1}^{\infty} [P_n(\cos\alpha) - P_n - 1(\cos\alpha)]z^n,$$

$$= 1 + \sum_{n=1}^{\infty} B_n z^n \qquad z \in \mathcal{U} \qquad (1.2)$$

By the geometric properties of Koebe function, the function ϕ maps onto the right plane

$\Re(w) > 0$ minus the slit along positive real axis from $\frac{1}{|cos\frac{\alpha}{2}|}$ to ∞.

The Koebe one-quarter theorem [5] states that the image of an analytic function $\mathcal{U}$ of every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$.

According to this, every function $f \in \mathcal{S}$ has an inverse function f^{-1}, which is defined by:

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U}),$$

and

$$f(f^{-1}(w)) = w \quad (w \in \mathcal{U}^{\square} = \{w \in \mathbb{C}: |w| < \tfrac{1}{4}\}),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots .$$

The function $f \in \mathcal{S}$ is said to be bi-univalent function if both f and f^{-1} are univalent in $\mathcal{U}$. Let σ be the class of all bi-univalent functions in $\mathcal{U}$ given by (1.1).

Lewin [8] is the first author introduced class of analytic bi-univalent function and also estimated the second coefficient $|a_2| < 1.51$. Many authors introduced several subclasses of analytic bi-univalent functions and also the bounds for first two coefficients $|a_2|$ and $|a_3|$. The problem of estimating coefficients $|a_n|, n \geq 2$ is still open problem.

Let Ω denote the class of all analytic functions w in $\mathcal{U}$ which satisfy the following conditions $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathcal{U}$

A function f is said to be subordinate to g, if there exist a Schwarz function $w \in \Omega$ such that $f(z) = g(w(z))$ and it is denoted as $f(z) \prec g(z)$.

If the function g is univalent in $\mathcal{U}$, then f is subordinate to g is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

The main reference for the following result is by the earlier work of Lashin, A.M.Y.; Badghaish, A.O.; Bajamal, A.Z [9]. They defined the class $L_\sigma(\gamma, \rho, \phi)$ as follows:

Definition 1 *A function* $f \in \sigma$ *given by (1.1) is said to be in the class* $L_\sigma(\gamma, \rho, \phi)$ *with* $0 \leq \gamma, \rho \leq 1$ *if the following subordinations are satisfied:*

$$(1 - \gamma + 2\rho)\frac{f(z)}{z} + (\gamma - 2\rho)f'(z) + \rho z f''(z) \prec \phi(z) \quad (z \in \mathcal{U}),$$

and

$$(1 - \gamma + 2\rho)\frac{g(w)}{w} + (\gamma - 2\rho)g'(w) + \rho w g''(w) \prec \phi(w) \quad (w \in \mathcal{U}),$$

where $g(w) = f^{-1}(w)$.

Lemma 1 *Let* $u, v \in \Omega$ *such that*

$$u(z) = \sum_{n=1}^{\infty} b_n z^n,$$

and

$$v(w) = \sum_{n=1}^{\infty} c_n w^n.$$

Then,

$$\phi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + (B_1 b_3 + 2b_1 b_2 B_2 + B_3 b_1^3) z^3 + \ldots\ldots.$$

and

$$\phi(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + (B_1 c_3 + 2c_1 c_2 B_2 + B_3 c_1^3) w^3 + \ldots..$$

Where,

$$B_1 = \cos\alpha - 1, B_2 = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha), B_3 = \frac{1}{2}(5\cos^3\alpha - 3\cos^2\alpha + 1).$$

2.Main results

Theorem 1 *Let the function* $f \in L_\sigma(\gamma, \rho, \phi)$*.Then,*

$$|a_2| \leq \frac{\sqrt{2}(1-\cos\alpha)}{\sqrt{2(1+2\gamma+2\rho)(1-\cos\alpha)+3(1+\gamma)^2(1+\cos\alpha)}}, \text{(2.1)}$$

and

$$|a_3| \leq [1 - \frac{(1+\gamma)^2}{(1+2\gamma+2\rho)|B_1|}]|a_2|^2 + \frac{|B_1|}{(1+2\gamma+2\rho)}. \text{(2.2)}$$

and for $z \in \mathcal{U}$,

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{|\cos\alpha - 1|}{2(1+2\gamma+2\rho)} & if|\xi - 1| \leq \frac{(\cos\alpha - 1)}{(\cos\alpha-1)-2(1+\gamma)^2(1+3\cos\alpha)} \\ |\cos\alpha - 1| & if|\xi - 1| \geq \frac{(\cos\alpha-1)}{(\cos\alpha-1)-2(1+\gamma)^2(1+3\cos\alpha)} \end{cases}$$

Proof. Let $f \in L_\sigma(\gamma, \rho, \phi)$ then,

$$|a_3 - \xi a_2^2| \leq \left| \frac{B_1}{2(1+2\gamma+2\rho)}(b_2 - c_2) - \xi \frac{B_1^3}{2[(1+2\gamma+2\rho)B_1^2 - (1+\gamma)^2 B_2}(b_2 + c_2) \right| \quad (2.3)$$

By expanding and re-arranging we get,

$$|a_3 - \xi a_2^2| \leq \frac{B_1}{2(1+2\gamma+2\rho)} \left[\left(\Omega - \xi \frac{B_1^2}{B_1^2 - (1+\gamma)^2 B_2} \right) b_2 - \left(\Omega + \xi \frac{B_1^2}{B_1^2 - (1+\gamma)^2 B_2} \right) c_2 \right]$$

Applying lemma to the above equation and after simple calculations,we get

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{|\cos\alpha - 1|}{2(1+2\gamma+2\rho)} & if|\xi - 1| \leq \frac{(\cos\alpha - 1)}{(\cos\alpha-1)-2(1+\gamma)^2(1+3\cos\alpha)} \\ |\cos\alpha - 1| & if|\xi - 1| \geq \frac{(\cos\alpha-1)}{(\cos\alpha-1)-2(1+\gamma)^2(1+3\cos\alpha)}. \end{cases} \quad (2.4)$$

Which completes proof of Theorem 1.

3. Conclusion

In the present study, we have used the Legendre polynomials to define new subclasses of bi-univalent function. Moreover,we derived the Fekete-Szego inequality for the subclasses. In future, we can find the third and fourth hankel determinant for this subclasses.

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