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# Third Hankel determinant for a subclass of analytic function related to modified Nephroid Domain

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**Abstract:**

The aim of this current work is, by utilizing the precept of subordination a new subclass of univalent function related to modified Nephroid Domain which is defined in open unit disk and estimated the third Hankel determinants.

## 1 Introduction

Nephroid domain is the phrase Nephroid is derived from a Greek word Nephros because of this a kidney fashioned simple curve was first used for the Epicycloid by using proctor in 1878. It is an Algebraic curve of degree 6 and may be portraited by means of rolling a circle of Radius r, with outside circle of radius 2r on top it. Hence, Nephroid is an "Epicycloid". In addition, Huygens proved that the Nephroid is the catacaustic of circle, whilst the mild supply is at Infinity, we an observation published by way of him on 1690. The Nephroid can be additionally generated as the envelope of circles, focused on a given circle and tangent to one of the circles diameter, stated by way of wells in 1990. Let $\mathcal{A}$ denote the class of analytic functions f(z) of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

which are *analytic* in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent function. Let $\mathcal{S}$ be the subclass of a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$ of starlike functions in $\mathbb{D}$, if it satisfies the following inequality:

$$\mathcal{S}^* = f \in \mathcal{S} : z\frac{f'(z)}{f(z)} \prec \frac{1+z}{1-z}, \qquad z \in \mathbb{D}. \qquad (1.2)$$

To recall the principle of subordination between analytic functions, let $f(z)$ and $g(z)$ be analytic functions in $\mathbb{D}$. We say that the function $f(z)$ is *subordinate* to $g(z)$, such that $f(z) = g(w(z)) \ (z \in \mathbb{D})$. We denote the subordination by $f \prec g$ (or $f(z) \prec g(z),\ z \in \mathbb{D}$). In particular, if the function $g(z)$ is univalent in $\mathbb{D}$, the subordination is equivalent to the conditions $f(0) = g(0),\ f(\mathbb{D}) \subset g(\mathbb{D})$.

$$\mathcal{S}^*_{Ne} = f \in s : \frac{2(zf'(z))}{(f(z)-f(-z))} \prec 1 + z - \frac{1}{3}z^3 \qquad (1.3)$$

Recently Noor [11] introduced the class $\mathcal{S}^*_{Ne}$ which are associated with Nephroid domain. The $q^{th}$ Hankel determinant is defined as the coefficients functions yield information on regarding the properties of univalent function. Bansal [3] computed an estimate for the second coefficient of normalized univalent analytic function and this bounds provides the growth, distortion and covering theorems. De Branges [2] proved that if ratio of two bounded analytic function in $\mathbb{D}$ then the function is rational for a given natural numbers n,q and $a_1 = 1$ the Hankel determinants $H_{q,n}(f)$ of a function $f \epsilon \mathcal{A}$ is defined by means of the following determinant.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (n, q \in \mathbb{N} = 1,2,3\ldots). \qquad (1.4)$$

This determinant is considered by many authors [5] [3] [14]. For example, the rate of growth of $H_q(n)$ as n $\to \infty$ for functions f(z) given by (1.4) with bounded boundary is determined by Noor [11]. For some specfic values of q and n, the quantites $H_{2,1}(f)$ $=a_3 - a_2^2$ and $H_{2,2}(f)$=$a_2a_4 - a_3^2$ are known as second Hankel determinant respectively find the estimate on the second

Hankel determinant $H_{2,2}(f)$ is then finding the estimate on the third Hankel determinant. The third Hankel determinants is to calculate the initial coefficients second Hankel determinant using the triangle inequality.

$$|H_{3,1}(f)| = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \quad (1.5)$$

$$|H_{3,1}(f)| \leq \frac{377}{678} \simeq 0.597 \quad (1.6)$$

for the class $\mathcal{S}^*$ respectively the third Hankel determinants for the class of starlike functions proved as $|H_{3,1}| \leq \frac{377}{678}$.

$$\mathcal{S}^* = f \in \mathcal{S}: z\frac{f'(z)}{f(z)} \prec \frac{1+z}{1-z}, \qquad z \in \mathbb{D} \quad (1.7)$$

Basal [3] was the first mathematician who investigated the bounds of third order Hankel determinant for the $\mathcal{S}^*$. Zaprawa [20] using the same approach, several authors [11], [12], [13] and [14] published their articles regarding $|H_{3,1}(f)|$ for certain of analytic and univalent functions. The determinant has been considered by several authors in the literature. For example, Noor [11] studied about the hankel determinant of mean p valent functions also determined the rate of growth $H_q(n)$ as $n \to \infty$ for the functions $\mathcal{S}$ with a bounded boundary. The Hankel transform of an integer sequence and some of it properties were discussed by Kumar and Ravichandran [5]. one can easily observe that the Fekete-Szego functional in $H_2(1)$. Fekete-szego then further generalized the estimate for the $|a_3 - a_2^2|$. The bounds for the functional $|a_2a_4 - a_3^2|$, The determinant is the particular case of estimates the greatest value of the functional $|a_3 - a_2^2|$ for functions in $\mathcal{S}$ many researches like Arif et al [1], Kowalczyk et al [6], Liu et al [7], Mahmood et al [9], Noor [11], and Orhan et al [12] have studied in the subclass of univalent functions. The maximum value of $H_2(2)$ has been investigated by several authors. For instance the reader can see the work initated by [11] and [13]. In this paper, we obtain Hankel coefficient estimates for the functions in the above defined class.

## 2 Main Results

**Theorem 1** *Let* $f(z) \in \mathcal{S}_{Ne}^*$ *of the form(1.1) ,then*

$$|a_2| \leq \frac{1}{2}, \quad (2.1)$$

$$|a_3| \leq \frac{-3}{8}, \quad (2.2)$$

$$|a_4| \leq \frac{31}{144}, \quad (2.3)$$

$$|a_5| \leq \frac{1237}{1152}. \quad (2.4)$$

*Proof.* since $f \in \mathcal{S}_{Ne}^*$ there exists an analytic function $s(z)...,|s(z)| \leq 1$ and $s(0) = 0$ such that, $2(zf'(z))\frac{f(z)}{f(-z)} = 1 + w(z) - \frac{1}{3}(w(z))^3$

Denote $\Psi(s(z)) = 1 + w(z) - \frac{1}{3}(w(z))^3$, and $k(z) = 1 + c_1z + c_2z^2 + \ldots\ldots = \frac{1+w(z)}{1-w(z)}$

$$1 + (w(z)) - \frac{1}{3}(w(z))^3 = 1 + \frac{1}{2}c_1z + (\frac{1}{2}c_2 - \frac{1}{4}c_1^2z^2 + (\frac{1}{12}c_1^3 - \frac{1}{2}c_2c_1 + \frac{1}{2}c_3)z^3 + (\frac{1}{4}c_1^2c_2 - \frac{1}{2}c_3c_1 - \frac{1}{4}c_2^2 + \frac{1}{2}c_4)z^4 + \ldots\ldots$$

And other side(3.2),

$$2f'(z)\frac{f(z)}{f(-z)}$$
$$= 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3$$
$$+(-a_2^4 + 4a_2^2a_3 - 4a_2a_4 - a_3^2 + 4a_5)z^4 + \ldots.$$

$$f'(z)\frac{f(z)}{f(-z)}$$
$$= 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3$$
$$+(-a_2^4 + 4a_2^2a_3 - 4a_2a_4 - a_3^2 + 4a_5)z^4 + \ldots \qquad (2.5)$$

$$a_2 = \frac{1}{2}c_1 \qquad (2.6)$$

$$a_3 = \frac{-3}{8}c_2 \qquad (2.7)$$

$$a_4 = \frac{1}{12}c_1^3 - \frac{1}{2}c_2c_1 + \frac{1}{2}c_3 \qquad (2.8)$$

$$a_5 = \frac{1}{4}c_1^2c_2 - \frac{1}{2}c_3c_1 - \frac{1}{4}c_2^2 + \frac{1}{2}c_4 \qquad (2.9)$$

Now using (2.3) and (2.4),we get,

$$|a_2| \le 1|a_3| \le \frac{1}{2}. \qquad (2.10)$$

Rearrange the equation (2.5),we may write

$$|a_4| = |\frac{1}{12}c_1^3 - \frac{1}{2}c_2c_1 + \frac{1}{2}c_3|. \qquad (2.11)$$

using triangle inequality along with (2.1)(2.2),we get

$$|a_4| \le \frac{31}{144}. \qquad (2.12)$$

Now rearrange the (2.6),we may write

$$|a_5| = |\frac{1}{4}c_1^2c_2 - \frac{1}{2}c_3c_1 - \frac{1}{4}c_2^2 + \frac{1}{2}c_4| \qquad (2.13)$$

Application of triangle inequality (2.2),we get

$$|a_5| \le \frac{1237}{1152}. \qquad (2.14)$$

**Theorem 2** *Let $f(z) \in \mathcal{S}_{Ne}^*$ be the form(1.1).Then*

$$|a_3 - \lambda a_2^2| \le \frac{1-2\lambda}{2}, \lambda \le 0 \frac{1}{2}, 0 \le \lambda \le 1 \frac{2\lambda - 1}{2}, \lambda \ge 1.$$

*Proof.* since using (2.4)(2.5), we get

$$|a_3 - \lambda a_2^2| = \frac{1}{4}|c_2 - \lambda c_1^2|. |a_3 - \lambda a_2^2| = \frac{1}{4}|\frac{3}{4} + \lambda|.$$

**Theorem 3** *Let* $f(z) \in \mathcal{S}^*_{Ne}$ *be of the form(1.1).Then for* $\xi \in \mathcal{C}$, *we* have $|a_3 - \xi a_2^2| \leq \frac{1}{2} max 1. |2\xi - 1|$

*Proof.* since using (2.5) and (2.6). we get $|a_3 - \xi a_2^2| = \frac{1}{4} |\frac{3}{2} + \xi|$ applying 2.6.we get the required results.

**Theorem 4** *Let* $f(z) \in \mathcal{S}^*_{Ne}$ *be the form (1.1).then*

$$|a_3 - a_2^2| \leq \frac{1}{2} \quad (2.15)$$

*Proof.* since using (2.5) and (2.6) , we get $|a_3 - a_2^2| = \frac{5}{8} > \frac{1}{2}$.

**Theorem 5** *Let* $f(z) \in \mathcal{S}^*_{Ne}$ *be the form (1.1).Then*

$$|a_2 a_3 - a_4| \leq \frac{31}{144}. \quad (2.16)$$

*Proof.* Since using (2.5), (2.6) and (2.7) also rearranging term, we get $|a_2 a_3 - a_4| = |\frac{1}{72}(c_1^3) + \frac{1}{6}(c_1 c_2) - \frac{1}{6} c_3|$ applying (2.1) and (2.2) we get $|a_2 a_3 - a_4| \leq \frac{31}{144}$.

**Theorem 6** *Let*$f(z) \in \mathcal{S}^*_{Ne}$ *be the form (1.1).Then*

$$|a_2 a_4 - a_3^2| \leq \frac{19}{551}. \quad (2.17)$$

*Proof.* since using (2.5),(2.6) and (2.7), we get $|a_2 a_4 - a_3^2| \leq \frac{19}{551}$.

**Theorem 7** *Let* $f(z) \in \mathcal{S}^*_{Ne}$ *be the form (1.1).Then*

$$|H_3, 1(f)| \leq \frac{377}{678} \simeq 0.597. \quad (2.18)$$

*Proof.* since

$$H_3, 1(f) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2). \quad (2.19)$$

by applying (2.5),we obtain

$$|H_3, 1(f)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_4 - a_2 a_3| + |a_5||a_3 - a_2^2|. \quad (2.20)$$

$$|H_3, 1(f)| \leq \frac{377}{678} \simeq 0.597.$$

### 3 conclusion

In this paper, the coefficients $a_3, a_4, a_5$ and the third Hankel determinant for a certain class of analytic function related to Nephroid domain were presented. The results of this article will encourage other researches to work fourth Hankel determinant due to its novely in literature. We hope that this work motivate the researches to the fourth Hankel determinant for the functions in other classes of univalent functions.

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