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UPPER BOUNDS FOR HARMONIC COINVARIANT OF GRAPHS

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Abstract

In this paper, we concentrate some standard graph products to obtain the upper bounds for harmonic coinvariant. In addition, we obtain the exact value of harmonic invariant and its coinvariant for double graph of a given graph.

Keywords: Topological invariant, Harmonic invariant, Graph operation.

1 Introduction

A chemical graph is a graph whose vertices denote atoms and edges denote bonds between those atoms of any underlying chemical structure. Topological indices are introduced to measure the characters of chemical molecules. A *topological invariant* for a (chemical) graph G is a numerical quantity invariant under automorphism of G and it does not depend on the labelling or pictorial representation of the graph. It has been used for examing quantitative structure-property relationship (QSPR) and quantitative structure-activity relationships (QSAR) extensively in which the biological activity or other properties of molecules are correlated with their chemical structures, see [1,2,5]. In the current chemical literature, a large number of graph-based structure descriptors (topological indices) have been put forward, that all depend only on the degrees of the vertices of the molecular graph. More details on vertex-degree-based topological indices and on their comparative study can be found in [6, 7, 8, 9, 3, and 4]

The *first Zagreb invariant* $M_1(\mathcal{G})$ is the equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb invariant* $M_2(\mathcal{G})$ is the equal to the sum of the products of the degrees of pairs of adjacent vertices, that is, $M_1(\mathcal{G}) = \sum_{r\epsilon V(\mathcal{G})} DG_{\mathcal{G}}^2(r) = \sum_{rs\epsilon E(\mathcal{G})}(DG_{\mathcal{G}}(r) + DG_{\mathcal{G}}(s))$, $M_2(\mathcal{G}) = \sum_{rs\epsilon E(\mathcal{G})} DG_{\mathcal{G}}(r)DG_{\mathcal{G}}(s)$, where $DG_{\mathcal{G}}(r)$ is a degree of a vertex r in $\mathcal{G}$.

For a connected graph $\mathcal{G}$, the harmonic invariant H ($\mathcal{G}$) is defind as $H(\mathcal{G}) = \sum_{rs\epsilon E(\mathcal{G})} \frac{2}{DG_{\mathcal{G}}(r)+DG_{\mathcal{G}}(s)}$. Deng et al. [18] considered the relation between the harmonic invariant of a graph and its chromatic number. Zhong [11, 12, 13] gave the minimum and maximum values of the harmonic invariant for simple graphs, trees, unicyclic graphs and graphs and graphs with girth at least k(k≥3) and characterized the corresponding external graphs, respectively. Lv et al. [16, 17] established the relationship between the harmonic invariant of a graph and its matching number. Shwetha et al. [15] derived expression for the harmonic invariant of some operations of graphs.

The *first and second Zagreb coindices* were first introduced by Ashrafi et al. [20].

They are defined as follows: $\overline{M_1}(\mathcal{G}) = \sum_{rs\notin E(\mathcal{G})}(DG_{\mathcal{G}}(r) + DG_{\mathcal{G}}(s)$,

$\overline{M_2}(\mathcal{G}) = \sum_{rs\notin E(\mathcal{G})} DG_{\mathcal{G}}(r)DG_{\mathcal{G}}(s)$. In this sequence, the harmonic coinvariant of $\mathcal{G}$ is defined

as $\overline{H}(\mathcal{G}) = \sum_{rs\notin E(\mathcal{G})} \frac{2}{DG_{\mathcal{G}}(r)+DG_{\mathcal{G}}(s)}$.

Since graph operation place important role to study the infinity graphs which are derived from the smaller graphs, in this view, we obtain the upper bounds for the harmonic coindices of some graph operations such as edge corona product graph and Mycielskian graph. Finally, the values of harmonic invariant and its coinvariant of double graph of a given graph are obtained.

2 Main Results

We denote by Δ and δ the maximum vertex degrees of $\mathcal{G}$, respectively. The inverse degree invariant of $\mathcal{G}$, denoted by ID ($\mathcal{G}$) = $\sum_{r\epsilon V(\mathcal{G})} \frac{1}{DG_{\mathcal{G}}(r)}$.

Theorem 2.1. Let $\mathcal{G}$ be a connected graph with n vertices and p pendent vertices. They

$\bar{H}(\mathcal{G}) \leq \frac{p}{6}(4n - p - 7)$.

Proof. Assume that $\mathcal{G}$ has exactly one pendent vertex, say x and y is its unique neighbour. Then

$\bar{H}(\mathcal{G}) \leq \sum_{z\epsilon V(\mathcal{G})\backslash\{x,y\}} \frac{2}{DG_{\mathcal{G}}(z)+1} \leq \sum_{z\epsilon V(\mathcal{G})\backslash\{x,y\}} \frac{2}{3} = \frac{2}{3}(n-2)$.

Now, we assume that p≥2. One can observe that each pair of pendent vertices contribute to $\bar{H}(\mathcal{G})$ is 1. The total contribution of pendent vertices pairs to $\bar{H}(\mathcal{G})$ is $\frac{p(p-1)}{2}$. Let x be a pendent vertex of G and y is its unique neighbour. Then for any non-pendent vertex z in $\mathcal{G}$, the contribution of vertex pairs {x, z} to $\bar{H}(\mathcal{G})$ is $\frac{2(n-p-1)p}{1+DG_{\mathcal{G}}(z)}$. Since $DG_{\mathcal{G}}(z) \geq 2$ for any non-pendent vertex z in $\mathcal{G}$, we obtain

$\bar{H}(\mathcal{G}) \leq \frac{p(p-1)}{2} + \frac{2(n-p-1)p}{3} = \frac{p}{6}(4n - p - 7)$.

This completes the proof.

Lemma 2.2. *[19] Let f be a convex function on the interval I and* $x_1, x_2, x_3, \ldots x_n \in I$. *Then*

$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$, *with equality if and only if* $x_1 = x_2 = x_3 = \cdots = x_n$.

2.1 Edge corona product:

Hou and Shiu [1] introduced a kind of new graph operation, namely, edge corona product. The *edge corona product* $\mathcal{G} \bullet \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ is defined as the graph obtained by taking one copy of $\mathcal{G}$ and *p* copies of $\mathcal{H}$, and then joining two end vertices of the i^{th} edge of $\mathcal{G}$ to every vertex in the i^{th} copy of $\mathcal{H}$. In [1], the adjacency spectrum and Laplacian spectrum of edge corona product of $\mathcal{G}$ and $\mathcal{H}$ were presented in terms of the spectrum and Laplacian spectrum of $\mathcal{G}$ and $\mathcal{H}$, respectively.

Theorem 2.3*: Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be two graphs with* n_1, n_2 *vertices and* m_1, m_2 *edges, respectively.Then*

$$\bar{H}(\mathcal{G}_1 \bullet \mathcal{G}_2) \leq (n_2+1)\bar{H}(\mathcal{G}_1) + \frac{m_1}{4}\bar{H}(\mathcal{G}_2) + \frac{n_1 n_2}{n_2+1}(ID(\mathcal{G}_1) - 1) + \frac{2{n_1}^2 + m_1(n_2 m_1 - n_2 - 4)}{8} ID(\mathcal{G}_2) + \frac{m_1\left(n_2(n_2-1) + {n_1}^2(m_1-1)\right)}{16} + \frac{n_2({n_1}^2 - 2m_1) - m_1 m_2}{8}.$$

Proof. Let x_{ij} be the j^{th} vertex in the i^{th} copy of H, $i = 1,2,\ldots, m_1, j = 1,2, \ldots n_2$ and let

y_k Be the k^{th} in $\mathcal{G}_1$, $k = 1,2, \ldots n_1$. Also let x_j be the j^{th} vertex in $\mathcal{G}_2$.

By the definition of edge corona, for each vertex x_{ij}, we have $DG_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_{ij}) = DG_{\mathcal{G}_2}(x_j) + 2$, and for every vertex y_k in $\mathcal{G}_1$, $DG_{\mathcal{G}_1 \bullet \mathcal{G}_2}(y_k) = DG_{\mathcal{G}_1}(y_k) n_2 + DG_{\mathcal{G}_1}(y_k) =$
$(n_2+1) DG_{\mathcal{G}_1}(y_k)$.

Now, we consider the following four cases of nonadjacent vertex pairs in $\mathcal{G}_1 \bullet \mathcal{G}_2$.

Case 1: The nonadjacent vertex pairs $\{x_{ij}; x_{ih}\}$, $1 \leq i \leq m_1, 1 \leq j < h \leq n_2$, and it is assumed that $x_i x_h \notin E(\mathcal{G}_2)$.

$$C_1 = \sum_{i=1}^{m_1} \sum_{x_{ij}x_{ih} \notin E(\mathcal{G}_1 \bullet \mathcal{G}_2)} \frac{2}{DG_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_{ij}) + DG_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_{ih})}$$

$$=\sum_{i=1}^{m_1}\sum_{x_jx_h\notin E(\mathcal{G}_2)}\frac{2}{DG_{\mathcal{G}_2}(x_j)+DG_{\mathcal{G}_2}(x_h)+4}$$

By Jensen's inequality, we have $\frac{2}{DG_{\mathcal{G}_2}(x_j)+DG_{\mathcal{G}_2}(x_h)+4}\le\frac{1}{2\left(DG_{\mathcal{G}_2}(x_j)+DG_{\mathcal{G}_2}(x_h)\right)}+\frac{1}{8}$ with equality if and only if $DG_{\mathcal{G}_2}(x_j)+DG_{\mathcal{G}_2}(x_h)=4$.thus

$$C_1\le\sum_{i=1}^{m_1}\sum_{x_jx_h\notin E(\mathcal{G}_2)}\frac{1}{2\left(DG_{\mathcal{G}_2}(x_j)+DG_{\mathcal{G}_2}(x_h)\right)}+\frac{1}{8}$$
$$=\sum_{i=1}^{m_1}\left(\frac{\overline{H}(\mathcal{G}_2)}{4}+\frac{1}{8}\left(\frac{\mathrm{n}_2(\mathrm{n}_2-1)}{2}-m_2\right)\right)$$
$$=\frac{m_1}{4}\overline{H}(\mathcal{G}_2)+\frac{m_1\mathrm{n}_2(\mathrm{n}_2-1)}{16}-\frac{m_1m_2}{8}$$

Case 2: The nonadjacent vertex pairs $\{y_k,y_s\}$, $1\le k<s\le n_1$ and it is assumed that $y_ky_s\notin E(\mathcal{G}_1)$. Thus

$$C_2=\sum_{y_ky_s\notin E(\mathcal{G}_1\cdot\mathcal{G}_2)}\frac{2}{DG_{\mathcal{G}_1\cdot\mathcal{G}_2}(y_k)+DG_{\mathcal{G}_1\cdot\mathcal{G}_2}(y_s)}=\sum_{y_ky_s\notin E(\mathcal{G}_1)}\frac{2}{(n_2+1)\left(DG_{\mathcal{G}_1}(y_k)+DG_{\mathcal{G}_1}(y_s)\right)}$$
$$=(n_2+1)\sum_{y_ky_s\notin E(\mathcal{G}1)}\frac{2}{DG_{\mathcal{G}_1}(y_k)+DG_{\mathcal{G}_1}(y_s)}$$
$$=(n_2+1)\overline{H}(\mathcal{G}1)$$

Case 3: The nonadjacent vertex pairs $\{x_{ij},y_k\}$, $1\le i\le m_1$, $1\le j\le n_2$, $1\le k\le n_1$, and it is assumed that the i^{th}edgee_i, $1\le i\le m_1$in$\mathcal{G}_1$ does not pass throughy_k,

Note that each vertex y_k is adjacent to all vertices of $DG_{\mathcal{G}_1}(y_k)$ copies of $\mathcal{G}_2$, that is, each y_k is not adjacent to any vertex of $m_1-DG_{\mathcal{G}_1}(y_k)$ copies of $\mathcal{G}_2$. Hence

$$C_3=\sum_{k=1}^{n_1}(n_1-DG_{\mathcal{G}_1}(y_k))\sum_{j=1}^{n_2}\frac{2}{DG_{\mathcal{G}_2}(x_j)+2+(n_2+1)DG_{\mathcal{G}_1}(y_k)}$$

By Jensen's inequality, we obtain $\frac{1}{DG_{\mathcal{G}_2}(x_j)+2+(n_2+1)DG_{\mathcal{G}_1}(y_k)}\le\frac{1}{4DG_{\mathcal{G}_2}(x_j)+2}+\frac{1}{4(n_2+1)DG_{\mathcal{G}_1}(y_k)}$ with equality if and only if $DG_{\mathcal{G}_2}(x_j)+2=(n_2+1)DG_{\mathcal{G}_1}(y_k)$.Thus

$$C_3\le\frac{1}{2}\sum_{k=1}^{n_1}(n_1-DG_{\mathcal{G}_1}(y_k))\sum_{j=1}^{n_2}\left(\frac{1}{DG_{\mathcal{G}_2}(x_j)+2}+\frac{1}{(n_2+1)DG_{\mathcal{G}_1}(y_k)}\right)$$
$$\le\frac{1}{2}\sum_{k=1}^{n_1}(n_1-DG_{\mathcal{G}_1}(y_k))\sum_{j=1}^{n_2}\left(\frac{1}{4DG_{\mathcal{G}_2}(x_j)}+\frac{1}{8}+\frac{1}{(n_2+1)DG_{\mathcal{G}_1}(y_k)}\right)$$
$$=\frac{1}{2}\sum_{k=1}^{n_1}(n_1-DG_{\mathcal{G}_1}(y_k))\left(\frac{ID(\mathcal{G}_2)}{4}+\frac{n_2}{8}+\frac{n_2}{(n_2+1)D\mathcal{G}_1(y_k)}\right)$$
$$=\frac{(n_1^2-2m_1)}{4}ID_{(\mathcal{G}_2)}+\frac{n_1n_2}{n_2+1}(ID_{(\mathcal{G}_1)}-1)+\frac{n_2(n_1^2-2m_1)}{8}$$

Case 4: The nonadjacent vertex pairs $\{x_{ij},x_{lh}\}$, $1\le i<\ell\le m_1$, $1\le j,h\le n_2$

$$C_4=\sum_{x_{ij}x_{lh}\notin E(\mathcal{G}_1\cdot\mathcal{G}_2)}\frac{2}{DG_{\mathcal{G}_1\cdot\mathcal{G}_2}(x_{ij})+DG_{\mathcal{G}_1\cdot\mathcal{G}_2}(x_{lh})}$$

$$=\frac{m_1(m_1-1)}{2}\sum_{j=1}^{n_2}\sum_{h=1}^{n_2}\frac{2}{DG_{\mathcal{G}_2}(x_j)+DG_{\mathcal{G}_2}(x_h)+4}$$

$$\leq\frac{m_1(m_1-1)}{8}\sum_{j=1}^{n_2}\sum_{h=1}^{n_2}\left(\frac{2}{DG_{\mathcal{G}_2}(x_j)+DG_{\mathcal{G}_2}(x_h)}+\frac{1}{2}\right)$$

$$\leq\frac{m_1(m_1-1)}{8}\sum_{j=1}^{n_2}\sum_{h=1}^{n_2}\left(\frac{1}{2DG_{\mathcal{G}_2}(x_j)}+\frac{1}{2DG_{\mathcal{G}_2}(x_h)}+\frac{1}{2}\right)$$

$$=\frac{m_1(m_1-1)}{16}(2n_2ID(\mathcal{G}_2)+n_2^2)$$

From the above four cases of nonadjacent vertex pair, we can obtain the hdesired result. This completes the proof.

2.2 Mycielskian graph

In a search for triangle-free graphs with arbitrarilylarge chromatic number, Mycielski [2] developed on a interesting graph transformation as follows: Let $\mathcal{G}$ be a connected graph with vertex set V($\mathcal{G}$) = $\{v_1,v_2,\ldots,v_n\}$. The *Mycielskian graph* $\mu(\mathcal{G})$ of $\mathcal{G}$ contains$\mathcal{G}$ itself as an isomorphic subgraph, together with n+1 additional vertices: a vertex u_i corresponding to each vertex v_i of $\mathcal{G}$, and another vertex w. Each vertex $u_i$is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$.

Lemma 2.4: Let $\mathcal{G}$be a connected graph on n vertices and m edges, Then for each i = 1,2,…,n, we have $DG_{\mu(\mathcal{G})}(v_i) = 2DG_{(\mathcal{G})}(v_i)$, $DG_{\mu(\mathcal{G})}(u_i) = DG_{(\mathcal{G})}(v_i)+1$ and $DG_{\mu(\mathcal{G})}(w) = n$.

By the definition of Mycielskian graph, for each edge $v_iv_j$of, $\mathcal{G}$ the Mycielskian graph include two edges$u_iv_j\notin$and v_iu_jNow we compute the upper bounds for harmonic coinvariant of Mycielskian graph.

Theorem 2.5: Let $\mathcal{G}$ be a graph on n vertices and m edges. Then

$$\bar{H}(\mu(\mathcal{G}))\leq\left(\frac{n(n-1)-2m+6}{8}\right)\bar{H}(\mathcal{G})+\frac{m}{4}H(\mathcal{G})+\frac{5}{12}ID(\mathcal{G})+\frac{n(n-1)-2m}{4(\delta+1)}+\frac{(n(n-1)-2m)^2}{16}+\frac{n(n+2)}{2(n+1)}+\frac{m^2}{4}.$$

Proof: Let V($\mu(\mathcal{G})$) = $\{v_1,\ldots,v_n\}$and let V($\mu(\mathcal{G})$) = $\{v_1,\ldots,v_n,u_1,\ldots,u_n,w\}$. By the structure of Mycielskian graph, if $v_iv_j\notin E(\mathcal{G})$, then $v_iu_j\notin E(\mathcal{G})$ and $u_iv_j\notin E(\mathcal{G})$.

Now we consider the following cases of nonadjacent vertex pairs in ($\mathcal{G}$).

Case 1: The nonadjacent vertex pairs $\{v_i, v_j\}$ in ($\mathcal{G}$)

$$C_2=\sum_{v_iv_j\notin E(\mu(\mathcal{G}))}\frac{2}{\left(DG_{\mu(\mathcal{G})}(v_i)+DG_{\mu(\mathcal{G})}(v_j)\right)}$$

$$=\sum_{v_iv_j\notin E(\mathcal{G})}\frac{2}{2DG_{\mathcal{G}}(v_i)+2DG_{\mathcal{G}}(v_j)},\text{by Lemma 2.4}$$

$$=\frac{\bar{H}(\mathcal{G})}{2}$$

Case 2: The nonadjacent vertex pairs $\{u_i, uj\}$ in μ ($\mathcal{G}$).

Case 2.1: $u_iu_j\notin E(\mu(\mathcal{G}))$and $v_iv_j\notin E(\mathcal{G})$.

$$C'_2=\sum_{u_iu_j\notin E(\mu(\mathcal{G}))}\frac{2}{\left(DG_{\mu(\mathcal{G})}(u_i)+DG_{\mu(\mathcal{G})}(u_j)\right)}$$

,

$$=\sum_{\text{vivj}\notin E(\mathcal{G})}\frac{2}{DG_{\mathcal{G}}(v_i)+DG_{\mathcal{G}}(v_j)+2},\text{by Lemma 2.4.}$$

By Jensen's inequality, we obtain$C'_2\leq\frac{1}{4}\sum_{\text{vivj}\notin E(\mathcal{G})}\left(\frac{2}{DG_{\mathcal{G}}(v_i)+DG_{\mathcal{G}}(v_j)}+1\right)$ with equality if and only if $DG_{\mathcal{G}}(v_i)+DG_{\mathcal{G}}(v_j)=2$. Thus $C'_2\leq\frac{1}{4}\left(\bar{H}(\mathcal{G})+\left(\frac{n(n-1)}{2}-m\right)\right)$.

Case 2.2: $u_iu_j\notin E(\mu(\mathcal{G}))$and $v_iv_j\in E(\mathcal{G})$.

$$C''_2 = \sum_{u_i u_j \notin E(\mu(\mathcal{G}))} \frac{2}{\left(DG_{\mu(\mathcal{G})}(u_i) + DG_{\mu(\mathcal{G})}(u_j)\right)}$$

$$= \sum_{v_i v_j \in E(\mathcal{G})} \frac{2}{DG_{\mathcal{G}}(v_i)+DG_{\mathcal{G}}(v_j)+2} \quad \text{by Lemma 2.4.}$$

Apply Jensen's inequality, we have $C''_2 \leq \sum_{v_i v_j \in E(\mathcal{G})} \left(\frac{1}{2(DG_{\mathcal{G}}(v_i)+DG_{\mathcal{G}}(v_j))} + \frac{1}{4}\right) = \frac{1}{4}(H(\mathcal{G}) + m)$

With equality if and only if $DG_{\mathcal{G}}(v_i) + DG_{\mathcal{G}}(v_j) = 2$.

If $u_i u_j \notin E(\mu(\mathcal{G}))$, then there are m edges $v_i v_j \in E(\mathcal{G})$ and $\frac{n(n-1)}{2} - m$ nonadjacent vertex pair $\{VI, VJ\}$ in $\mathcal{G}$ as well as $\mu(\mathcal{G})$. By cases 2.1 and 2.2, we have the contribution of nonadjacent vertex pair of case 2 is given by

$$C_2 = \left(\frac{n(n-1)}{2} - m\right)C'_2 + mC''_2$$

$$= \frac{1}{4}\left(\frac{n(n-1)}{2} - m\right)\left(\bar{H}(\mathcal{G}) + \left(\frac{n(n-1)}{2} - m\right)\right) + \frac{m}{4}(H(\mathcal{G}) + m).$$

Case 3: The nonadjacent vertex pairs $\{u_i, v_i\}$ in $\mu(\mathcal{G})$ for each i = 1, 2,

$$C_3 = \sum_{i=1}^{n} \frac{2}{DG_{\mu(\mathcal{G})}(u_i)+DG_{\mu(\mathcal{G})}(v_i)}$$

$$= \sum_{i=1}^{n} \frac{2}{3DG_{(\mathcal{G})}(v_i)+1}, \text{ by lemma 2.4}$$

$$\leq \frac{1}{4}\sum_{i=1}^{n}\left(\frac{2}{3DG_{(\mathcal{G})}(v_i)} + 2\right) \quad \text{, by Jensen's inequality}$$

$$= \frac{1}{4}\left(\frac{2}{3}\, ID(\mathcal{G}) + 2n\right)$$

Case 4: The nonadjacent vertex pairs $\{u_i, v_j\}$ in $\mu(\mathcal{G})$.

$$C_4 = \sum_{u_i v_j \in E(\mu(\mathcal{G}))} \frac{2}{DG_{\mu(\mathcal{G})}(u_i)+DG_{\mu(\mathcal{G})}(v_j)}$$

$$= \sum_{v_i v_j \in E(\mathcal{G})} \frac{2}{DG_{(\mathcal{G})}(v_i)+2DG_{(\mathcal{G})}(v_j)+1}, \text{ by lemma 2.4}$$

$$\leq \frac{1}{4}\sum_{v_i v_j \in E(\mathcal{G})} \frac{2}{DG_{(\mathcal{G})}(v_i)+DG_{(\mathcal{G})}(v_j)} + \frac{1}{4}\sum_{v_i v_j \in E(\mathcal{G})} \frac{2}{DG_{(\mathcal{G})}(v_j)+1}$$

$$= \frac{\bar{H}(G)}{4} + \frac{n(n-1)-2m}{4(\delta+1)}$$

Case 5: The nonadjacent vertex pairs $\{w, v_i\}$ in $\mu(\mathcal{G})$ for each i = 1, 2, n.

$$C_5 = \sum_{v_i w \notin E(\mu(\mathcal{G}))} \frac{2}{DG_{\mu(\mathcal{G})}(V_i) + DG_{\mu(\mathcal{G})}(W)}$$

$$= \sum_{v_i \in V(\mathcal{G})} \frac{2}{2DG_{\mathcal{G}}(V_i)+(n+1)}$$

$$\leq \frac{1}{4}\sum_{v_i \in V(\mathcal{G})}\left(\frac{1}{DG_{\mathcal{G}}(V_i)} + \frac{2}{n+1}\right)$$

$$= \frac{1}{4}\left(ID(\mathcal{G}) + \frac{2n}{n+1}\right).$$

From the above five cases of nonadjacent vertex pairs, we can obtain the desired results. This completes the proof.

2.3 Double graph

Let $\mathcal{G}$ be a graph with V ($\mathcal{G}$) = $\{v_1, v_2, \dots, v_n\}$. The vertices of the double graph $\mathcal{G}^*$ are given by the two sets X = $\{x_1, x_2, \dots, x_n\}$ and Y = $\{y_1, y_2, \dots, y_n\}$. Thus for each vertex $v_i \in$V ($\mathcal{G}$), there are two vertices x_i and y_i in V ($\mathcal{G}^*$). The double graph $\mathcal{G}^*$ includes the initial edge set of each copies of, and for any edge $v_i\, v_j \in$E ($\mathcal{G}$), two more edges $x_i\, y_j$ and $x_j y_i$ are added. For a given vertex v in $\mathcal{G}$, let $D_{\mathcal{G}}(V) = \sum_{uv \notin E(\mathcal{G})} \frac{2}{DG_{\mathcal{G}}(u) + DG_{\mathcal{G}}(v)}$.

Theorem 2.6. The harmonic invariant of the double graph $\mathcal{G}^*$ of a graph $\mathcal{G}$ is given by

H ($\mathcal{G}^*$) = 8 IS I ($\mathcal{G}$).

Proof. From the definition of double graph it is clear that $DG_{\mathcal{G}^*}(x_i)$= $DG_{\mathcal{G}^*}(y_i)$=$2DG_{\mathcal{G}}(v_i)$, where $v_i \in$V ($\mathcal{G}$) and $x_i, y_i \in$V ($\mathcal{G}^*$) are corresponding clone vertices of v_i. Thus

$$H(\mathcal{G}^*) = \sum_{uv \notin E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}^*}(u) + DG_{\mathcal{G}^*}(v)}$$

$$= \sum_{x_i\, x_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}^*}(x_i) + DG_{\mathcal{G}^*}(x_j)} + \sum_{y_i\, y_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}^*}(y_i) + DG_{\mathcal{G}^*}(y_j)}$$

$$+ \sum_{x_i\, y_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}^*}(x_i) + DG_{\mathcal{G}^*}(y_j)} + \sum_{x_j y_i \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}}(x_j) + DG_{\mathcal{G}^*}(y_j)}$$

$$= 4 \sum_{v_i v_j \in E(\mathcal{G})} \frac{2}{2DG_{\mathcal{G}}(V_i) + 2DG_{\mathcal{G}}(V_j)} = 2\,H(\mathcal{G})$$

Theorem 2.7. Let $\mathcal{G}$ be a connected graph with n vertices and m edges. Then $\overline{H}(\mathcal{G}^*) = 2\,\overline{H}(\mathcal{G}) + \frac{ID(\mathcal{G})}{2}$.

Proof. Let V ($\mathcal{G}$) = $\{v_1, v_2, \dots, v_n\}$. Suppose that x_i and y_i are the corresponding clone vertices, in $\mathcal{G}^*$, of v_i for each $i = 1,2,\dots n$ for any given vertex v_i in $\mathcal{G}$ and its clone vertices x_i and y_i, there exists $DG_{\mathcal{G}^*}(x_i) = DG_{\mathcal{G}^*}(y_i) = 2DG_{\mathcal{G}}(v_i)$ by the definition of double graph.

For $v_i, v_j \in$ V($\mathcal{G}$), if $v_i v_j \notin$ E($\mathcal{G}$), then $x_i x_j \notin$ E($\mathcal{G}$), $y_i y_j \notin$ E($\mathcal{G}$), $x_i y_j \notin$ E($\mathcal{G}$) and $y_i x_j \notin$ E($\mathcal{G}$).

So we need only to consider total contribution of the following three types of nonadjacent vertex pairs to calculate $\overline{H}(\mathcal{G})$.

Case 1: The nonadjacent vertex pairs $\{x_i, x_j\}$ and $\{y_i, y_j\}$, Where $v_i\, v_j \notin$E($\mathcal{G}$).

$$\sum_{y_i y_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}^*}(y_i) + DG_{\mathcal{G}^*}(y_j)} = \sum_{x_i x_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}^*}(x_i) + DG_{\mathcal{G}^*}(x_j)}$$

$$= \sum_{v_i v_j \notin E(\mathcal{G})} \frac{2}{2DG_{\mathcal{G}}(v_i) + 2DG_{\mathcal{G}}(v_j)}$$

$$= \frac{\overline{H}(\mathcal{G})}{2}$$

Case 2: The nonadjacent vertex pairs $\{x_i y_i\}$ *for each* $i = 1,2,3 \dots n$

$$\sum_{i=1}^{n} \frac{2}{DG_{\mathcal{G}*}(x_i) + DG_{\mathcal{G}*}(y_I)} = \sum_{i=1}^{n} \frac{2}{2DG_{\mathcal{G}}(v_i) + 2DG_{\mathcal{G}}(v_i)} = \sum_{i=1}^{n} \frac{1}{2DG_{\mathcal{G}}(v_i)} = \frac{ID(\mathcal{G})}{2}$$

Case 3: The nonadjacent vertex pairs $\{x_i, y_j\}$ and $\{y_j, x_i\}$, Where $v_i v_j \notin$E($\mathcal{G}$).

For each x_i, *there exist* n-1-$DG_{\mathcal{G}}(v_i)$, vertices in the set $\{y_1, y_2, \dots \dots y_n\}$, among which every vertex together with x_i compose a nonadjacent vertex pairs of $\mathcal{G}$*. The total contribution of these n-1-$DG_{\mathcal{G}}(v_i)$ nonadjacent vertex to calculate $\overline{H}(\mathcal{G}*)$ is

$$\sum_{x_i y_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}*}(x_i) + DG_{\mathcal{G}*}(y_j)} = \sum_{v_i v_j \in E(\mathcal{G}^*)} \frac{2}{2DG_{\mathcal{G}}(v_i) + 2DG_{\mathcal{G}}(v_j)} = \frac{D\mathcal{G}(^{v_i})}{2}$$

Hence

$$\sum_{i\neq j, x_i y_j \notin E(\mathcal{G}*)} \frac{2}{DG_{\mathcal{G}*}(x_i) + DG_{\mathcal{G}*}(y_j)} = \sum_{i=1}^{n} \frac{D_{\mathcal{G}(v_i)}}{2} = \overline{H}(\mathcal{G})$$

Hence

$$\overline{H}(\mathcal{G}^*) = \sum_{x_i x_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}*}(x_i) + DG_{\mathcal{G}*}(x_j)} + \sum_{y_i y_j \in E(\mathcal{G}^*)} \frac{2}{DG_{\mathcal{G}*}(y_i) + DG_{\mathcal{G}*}(y_j)}$$

$$+\sum_{i=1}^{n} \frac{2}{DG_{\mathcal{G}*}(x_i)+DG_{\mathcal{G}*}(y_j)} + \sum_{i\neq j, x_i y_j \notin E(\mathcal{G}*)} \frac{2}{DG_{\mathcal{G}*}(x_i)+DG_{\mathcal{G}*}(y_j)}$$

$$=2\overline{H}(\mathcal{G}) + \frac{ID(\mathcal{G})}{2}$$

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