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Some Fixed Point Results in M-fuzzy *b*-metric Space

Happy Hooda[1], Manjeet Rathee [2] & Archana Malik[3]

[1]Research Scholar, Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, (Hr) INDIA

[2]Assistant Professor, All India Jat Heros' Memorial College Rohtak-124001, (Hr) INDIA

[3]Professor, Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, (Hr) INDIA

Abstract: In this paper, we define M-fuzzy *b*-metric space and then we state and prove several fixed point theorems in M-fuzzy *b*-metric space. We also furnish a sufficient condition for a sequence to be Cauchy in M-fuzzy *b*-metric space. These theorems generalize and improve some known fixed point theorems.

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1. Introduction and Preliminaries:

The foundation of fuzzy mathematics is laid by L. A. Zadeh [21] in 1965. Fixed point theory is considered to be the most interesting and dynamic area of research and development of nonlinear analysis. Heilpern [9] first introduced the concept of fuzzy contractive mappings and a fixed point theorem for these mappings in metric linear spaces. The concept of fuzzy set was used in topology and analysis by many authors. George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [14]. This can be regarded as a generalization of statistical metric space. Banach contraction principal [2] is an initiative for researchers during last few decades. This principal plays an important role in investigating the existence and uniqueness of solution to various problems in mathematics which leads to mathematical models design by system of nonlinear integral equations, functional equations and differential equations. Banach contraction principal has been generalized in different directions by changing the condition of contraction. In addition to fuzzy metric space, there are still many extensions of metric and metric space terms. Bakhtin [1] and Czerwik [3] introduced a space where, instead of triangle inequality, a weaker condition was observed, with the aim of generalization of Banach contraction principal [2]. They called these spaces *b*-metric spaces. Many authors [6, 15, 18, 20] have proved fixed point theorems in fuzzy metric spaces. One such generalization is generalized metric space and D-metric space initiated by Dhage [4] in 1992. In 2006, Sedghi and Shobe [19] introduced D*-metric space as a probable modification of D-metric space and studied some topological properties which are not valid in D-metric spaces [19]. Based on D∗- metric concepts, they [19] define M-fuzzy metric space and proved a common fixed point theorem for two mappings under the conditions of weak compatible and R-weakly commuting mappings in complete M-fuzzy metric spaces. In addition to fuzzy metric spaces, there are still many extensions of metric and metric space terms.

Bakhtin[1] and Czerwik[3] introduced a space where, instead of triangle inequality, a weaker condition was observed, with the aim of generalization of Banach contraction principal [2].They called these spaces *b*-metric spaces. Relation between *b*-metric and fuzzy metric spaces is considering in [8]. On the other hand, in [16] the notion of a fuzzy *b*-metric space was introduced, where the triangle inequality is replaced by a weaker one. Now, in this paper we introduced a new space that is M-fuzzy *b*-metric space with the help of M-fuzzy metric space and *b*-Metric space and then we prove several fixed point theorems in M-fuzzy *b*-metric space. These theorems generalize and improve some known fixed point theorems in literature.

Definition 1.1:- A binary operation $T:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) T is associative and commutative,

(2) T is continuous,

(3) $T(a,1)=a$ for all $a\in[0,1]$,

(4) $T(a,b)\le T(c,d)$ for $a,b,c,d\in[0,1]$ such that $a\le c$ and $b\le d$.

Typical example of a continuous *t*-norm are $T_p(a,b)=ab$, $T_{\min}(a,b)=\min\{a,b\}$ and $T_L(a,b)=\max\{a+b-1,0\}$.

Definition 1.2:- Let T be a *t*-norm, and let $T_n:[0,1]\to[0,1]$, $n\in ¥$, be defined in the following way: $T_1(x)=T(x,x)$, $T_{n+1}(x)=T(T_n(x),x)$, $n\in ¥$, $x\in[0,1]$. We say that a *t*-norm of *H*-type if the family $\{T_n(x)\}_{n\in ¥}$ is equi-continuous at $x=1$. A trivial example of *t*-norm of H-type is $T_{\min}$;

Each t-norm of H-type can be extended by associatively in a unique way to an n-ary Operation taking for $(x_1,.....,x_n)\in[0,1]^n$ the values

$$T_{i=1}^1 x_i = x_1, \qquad T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, ..., x_n).$$

Example 1.3:- n-ary extensions of the t-norm $T_{\min}$, T_L and T_p are the followings:

$$T_{\min}(x_1,.......,x_n) = \min(x_1,.......,x_n),$$
$$T_L(x_1,.......,x_n) = \max(\sum_{i=1}^n x_i - (n-1), 0),$$
$$T_p(x_1,.......,x_n) = \prod_{i=1}^n x_i.$$

A t-norm T can be extended to a countable infinite operation taking for any sequence $(x_n)_{n\in¥}$ from $[0,1]$ the values $T_{i=1}^\infty x_i = \lim_{n\to\infty} T_{i=1}^n x_i$. The sequence $(T_{i=1}^n x_i)_{n\in¥}$ is non-increasing and bounded from below, and hence the limit $T_{i=1}^\infty x_i$ exists.

In the fixed point theory, it is of interest to investigate the classes of t-norms T and sequences (x_n) from the interval [0,1] such that

$$\lim_{n\to\infty} x_n = 1 \ and \ \lim_{n\to\infty} T_{i=1}^\infty x_i = \lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1. \qquad (1.1)$$

Proposition 1.4:- Let $(x_n)_{n\in¥}$ be a sequences of numbers from [0, 1] such that $\lim_{n\to\infty} x_n = 1$,

And let T be a t-norm of H-type. Then $\lim_{n\to\infty} T_{i=1}^\infty x_i = \lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1.$

Definition 1.5:- The 3-tuple (X, M, T) is known as fuzzy metric space (shortly, FM-space) if X is an any set, T is a continuous t-norm, and M is a fuzzy set in $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$;

(FM-1) $M(x,y,t)>0,$

(FM-2) $M(x,y,t)=1$ if and only if $x=y,$

(FM-3) $M(x,y,t)=M(y,x,t),$

(FM-4) $T(M(x,y,t),M(y,z,s))\le M(x,z,t+s),$

(FM-5) $M(x,y,g):[0,\infty)\to[0,1]$ is continuous.

Definition 1.6:- A 3-tuple $(X, M, *)$ is called a M-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^3\times(0,\infty)$, satisfying the following conditions for each $x,y,z,a\in X$ and $s,t>0$;

(M1) $M(x,y,z,t)>0,$

(M2) $M(x,y,z,t)=1$ if and only if $x=y=z.$

(M3) $M(x,y,z,t)=M(\{x,y,z\},t),$ (Symmetry) where p is a permutation function,

(M4) $M(x,y,a,t)*M(a,z,z,s)\le M(x,y,z,t+s),$

(M5) $M(x,y,z,g):(0,\infty)\to[0,1]$ is continuous.

Definition 1.7:- The 3-tuple (X, M, T) is known as fuzzy b-metric space if X is any set, T is a continuous t-norm, and M is a fuzzy set in $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0,$ and a given real number $b\ge 1,$

(BM-1) $M(x,y,t)>0,$

(BM-2) $M(x,y,t)=1$ if and only if $x=y,$

(BM-3) $M(x,y,t)=M(y,x,t),$

(BM-4) $T(M(x,y,\frac{t}{b}),M(y,z,\frac{s}{b}))\le M(x,z,t+s),$

(BM-5) $M(x,y,g):[0,\infty)\to[0,1]$ is continuous.

Definition 1.8:- The 3-tuple (X,M,T) is known as M-fuzzy *b*-metric space if *X* is any set, *T* is a continuous *t*-norm, and *M* is a fuzzy set in $X\times X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z,a\in X$ and *s*, *t* > 0 and a given real number $b\geq 1$, (MF-1) $M(x,y,z,t)>0$,
(MF-2) $M(x,y,z,t)=1$ if and only if $x=y=z$. (MF-3)
$M(x,y,z,t)=M(\{x,y,z\},t)$, (Symmetry) where *p* is a permutation function, (MF-4)
$T(M(x,y,a,\frac{t}{b}),M(a,z,z,\frac{s}{b}))\leq M(x,y,z,t+s)$, (MF-5) $M(x,y,z,\cdot):(0,\infty)\to[0,1]$ is continuous.

Definition 1.9:- A function $f:\mathbb{R}\to\mathbb{R}$ is called *b*-non-decreasing if $x>by$ implies $f(x)\geq f(y)$ *for all* $x,y\in\mathbb{R}$.

Lemma 1.10:- Let $(X,M,\cdot)$ be M-fuzzy *b*-metric space. Then $M(x,y,z,t)$ is *b*-non-decreasing with respect to *t* for all $x,y,z\in X$.

Definition 1.11:- Let (X,M,T) be M-fuzzy *b*-metric space. For $t>0$, the open ball $B(x,y,r,t)$ with center $x\in X$ and radius $0<r<1$ is defined as $B(x,y,r,t)=\{z\in X:M(x,y,z,t)>1-r\}$. A sequence $\{x_n\}$:

(a) Converges to *x* if $M(x_n,x,y,t)\to 1\quad as\quad n\to\infty\quad for\, each\; t>0$. In this case, we write $\lim_{n\to\infty}x_n=x$;

(b) Is called a Cauchy sequences if for all $0<\varepsilon<1\;and\;t>0$, there exist $n_0\in\mathbb{N}$ such that $M(x_n,x_m,y,t)>1-\varepsilon\quad for\,all\quad n,m\geq n_0$.

Definition 1.12:- The M-fuzzy *b*-metric space (X,M,T) is said to be complete if every Cauchy sequence is convergent.

Lemma 1.13:- In M-fuzzy *b*-metric space (X,M,T) we have:

(i) If a Sequence $\{x_n\}\;in\;X$ converges to *x*, then *x* is unique,

(ii) If a Sequence $\{x_n\}\;in\;X$ converges to *x*, then it is a Cauchy sequence.

Proposition 1.14:- Let (X,M,T) M-fuzzy *b*-metric space and suppose that $\{x_n\}$ converges to *x*. Then we have

$$M(x,y,z,\tfrac{t}{b})\leq\lim_{n\to\infty}\sup M(x_n,y,z,t)=M(x_n,y,z,bt),$$
$$M(x,y,z,\tfrac{t}{b})\leq\lim_{n\to\infty}\inf M(x_n,y,z,t)=M(x_n,y,z,bt).$$

2. Main Result:

Lemma 2.1:- Let $\{x_n\}$ be a sequence in M-fuzzy *b*-metric space (X,M,T). Suppose that there exists $\lambda\in(0,\frac{1}{b})$ such that

$$M(x_n,x_{n+1},x_{n+2},t)\geq M\left(x_{n-1},x_n,x_{n+1},\frac{t}{\lambda}\right),\quad n\in\mathbb{N},\;t>0, \tag{2.1}$$

And there exists $x_0,x_1,x_2\in X\;and\quad v\in(0,1)$ such that

$$\lim_{t\to\infty}T_{i=n}^{\infty}M\left(x_0,x_1,x_2,\frac{t}{v^i}\right)=1,\qquad t>0, \tag{2.2}$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof: Let $\sigma \in (\lambda b, a)$. Then the sum $\sum_{i=1}^{\infty} \sigma^i$ is convergent, and there exists $n_0 \in ¥$ such that $\sum_{i=1}^{\infty} \sigma^i < 1$ for every $n > n_0$. Let $n > m > n_0$. Since *M* is a *b*-nondecreasing, by (MF4), for every $t > 0$, we have

$$M(x_n, x_{n+m}, x_{n+m+1}, t) \geq M\left(x_n, x_{n+m}, x_{n+m+1}, \frac{t\sum_{i=n}^{n+m}\sigma^i}{b}\right)$$

$$\geq T\left(M\left(x_n, x_{n+1}, x_{n+2}, \frac{t\sigma^n}{b^2}\right), M\left(x_{n+1}, x_{n+m}, x_{n+m+1}, \frac{t\sum_{i=n+1}^{n+m}\sigma^i}{b^2}\right)\right)$$

$$\geq T\left(M\left(x_n, x_{n+1}, x_{n+2}, \frac{t\sigma^n}{b^2}\right), T\left(M\left(x_{n+1}, x_{n+2}, x_{n+3}, \frac{t\sigma^{n+1}}{b^3}\right),\right.\right.$$

$$\left.\left. \ldots, M\left(x_{n+m-1}, x_{n+m}, x_{n+m+1}, \frac{t\sigma^{n+m-1}}{b^m}\right)\ldots\right)\right).$$

By (2.1) if follows that

$$M(x_n, x_{n+1}, x_{n+2}, t) \geq M\left(x_0, x_1, x_2, \frac{t}{\lambda^n}\right), \qquad n \in ¥,\ t > 0,$$

And since $n > m$ and $b > 1$, we have

$$M(x_n, x_{n+m}, x_{n+m+1}, t) \geq T\left(M\left(x_0, x_1, x_2, \frac{t\sigma^n}{b^2\lambda^n}\right), T\left(M\left(x_0, x_1, x_2, \frac{t\sigma^{n+1}}{b^3\lambda^{n+1}}\right), \ldots M\left(x_0, x_1, x_2, \frac{t\sigma^{n+m}}{b^{m+1}\lambda^{n+m}}\right)\ldots\right)\right)$$

$$\geq T_{i=n}^{n+m} M\left(x_0, x_1, x_2, \frac{t\sigma^i}{b^{i-n+2}\lambda^i}\right)$$

$$\geq T_{i=n}^{n+m} M\left(x_0, x_1, x_2, \frac{t\sigma^i}{b^i\lambda^i}\right)$$

$$\geq T_{i=n}^{\infty} M\left(x_0, x_1, x_2, \frac{t}{\nu^i}\right),$$

where $\nu = \frac{b\lambda}{\sigma}$. Since $\nu \in (0,1)$, by (2.2) it follows that $\{x_n\}$ is Cauchy sequence.

By Proposition 1.3 the next corollary immediately follows.

Corollary 2.2:- Let $\{x_n\}$ be a sequence in a M-fuzzy *b*-metric space (X, M, T), and let T be of *H*-type. If there exists $\lambda \in \left(0, \frac{1}{b}\right)$ such that

$$M(x_n, x_{n+1}, x_{n+2}, t) \geq M\left(x_{n-1}, x_n, x_{n+1}, \frac{t}{\lambda}\right), \quad n \in ¥,\ t > 0, \tag{2.3}$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.3:- If for some $\lambda \in (0,1)$ and $x, y \in X$,

$$M(x, y, z, t) \geq M\left(x, y, z, \frac{t}{\lambda}\right), \quad t > 0, \tag{2.4}$$

then $x = y = z$.

$$M(x,y,z,t) \geq M\left(x,y,z,\frac{t}{\lambda^n}\right), \qquad t>0, \quad n\in ¥, t>0.$$

Proof: Condition (2.4) implies that

Now, $$M(x,y,z,t) \geq \lim_{n\to\infty} M\left(x,y,z,\frac{t}{\lambda^n}\right)=1,\ t>0,$$ And by (MF1) it follows that $x=y=z.$ **Theorem 2.4:-** Let (X,M,T) be a complete M-fuzzy *b*-metric space, and let $f: X\to X.$ Suppose that there exists $\lambda \in \left(0,\frac{1}{b}\right)$ such that

$$M(fx, fy, fz,t) \geq M\left(x,y,z,\frac{t}{\lambda}\right), \qquad x,y,z\in X, t>0, \tag{2.5}$$

And hence there exists $x_0 \in X$ and $\nu \in (0,1)$ such that

$$\lim_{n\to\infty} T_{i=n}^{\infty} M\left(x_0, fx_0, fx_0, \frac{t}{\nu^i}\right)=1, \qquad t>0. \tag{2.6}$$

Then f has a fixed point in X .

Proof: Let $x_0 \in X$ and $x_{n+1}=fx_n,\ n\in ¥.$ If we take $x=x_n, y=x_{n-1}$ and $z=x_{n-2}$ in (2.5), then we have

$$M\left(x_n, x_{n+1}, x_{n+2}, t\right) \geq M\left(x_{n-1}, x_n, x_{n+1}, \frac{t}{\lambda}\right), \qquad n\in ¥, t>0,$$

and by Lemma 2.1,it follows that $\{x_n\}$ is a Cauchy sequence. Since (X,M,T) is a complete, there exists $x\in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} M(x,x,x_n,t)=1, \quad t>0.$

(2.7) Conditions (2.5) and (MF4) are used to show that x is a fixed point for f :

$$M(fx,x,x,t) \geq T\left(M\left(fx,x_n,x,\frac{t}{2b}\right), M\left(x_n,x,x,\frac{t}{2b}\right)\right)$$
$$\geq T\left(M\left(x,x_{n-1},x_{n-2},\frac{t}{2b\lambda}\right), M\left(x_n,x,x,\frac{t}{2b}\right)\right)$$

for all $t>0.$ By (2.7), as $n\to\infty,$ we get $M\left(fx,x,x,t\right)\geq T(1,1)=1.$ Suppose that x, y and z are fixed points for f . By (2.5) we have

$$M\left(x,y,z,t\right)=M\left(fx,fy,fz,t\right) \geq M\left(x,y,z,\frac{t}{\lambda}\right), \qquad t>0,$$

and Lemma 2.3 implies that $x=y=z.$

Theorem 2.5:- Let (X,M,T) be a complete M-fuzzy *b*-metric space, and let $f: X\to X$. Suppose that there exists $\lambda\in\left(0,\frac{1}{b}\right)$ such that

$$M(fx,fy,fz,t) \geq \min\left\{M\left(x,y,z,\frac{t}{\lambda}\right), M\left(fx,x,z,\frac{t}{\lambda}\right), M\left(fy,y,z,\frac{t}{\lambda}\right), M\left(y,fz,z,\frac{t}{\lambda}\right), M\left(x,fz,z,\frac{t}{\lambda}\right)\right\}, \tag{2.8}$$

For all $x,y,z\in X, t>0,$ and there exists $x_0\in X$ *and* $v\in(0,1)$ such that $\lim_{n\to\infty} T_{i=n}^{\infty} M\left(x_0, fx_0, fx_0, \frac{t}{v^i}\right)=1, t>0.$ (2.9)

Then f has a unique fixed point in X.

Proof: Let $x_0 \in X$ and $x_{n+1} = fx_n,\ n \in ¥$. By (2.8) with $x = x_n,\ y = x_{n-1}$ and $z = x_{n-2}$, for every $n \in ¥$ and every $t > 0$,

$$M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq \min\left\{M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{\lambda}\right), M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{\lambda}\right), M\left(x_n, x_n, x_n, \frac{t}{\lambda}\right)\right\}$$

$$\geq \min\left\{M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{\lambda}\right), M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{\lambda}\right)\right\}$$

we have

If $M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{\lambda}\right),\ n \in ¥,\ t > 0,$ Then by Lemma 2.3 it follows that $x_{n-1} = x_n = x_{n+1},\ n \in ¥$.

So, $M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{\lambda}\right), \quad n \in ¥,\ t > 0,$

And by Lemma 2.1 we have $\{x_n\}$ is a Cauchy sequence. Hence there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} M\left(x, x_n, x, t\right) = 1, \quad t > 0. \qquad (2.10)$$

Let us prove that x is a fixed point for f. Let $\sigma_1 \in (\lambda b, 1)$ and $\sigma_2 = 1 - \sigma_1$. By (2.8) we get

$$M\left(fx, fy, fz, t\right) \geq \min\left\{M\left(x, y, fz, \frac{t}{\lambda}\right), M\left(x, fx, z, \frac{t}{\lambda}\right), M\left(y, fy, z, \frac{t}{\lambda}\right)\right\}$$

$$\geq \min\left\{M\left(x, y, z, \frac{t}{\lambda}\right), 1, 1\right\} = M\left(x, y, z, \frac{t}{\lambda}\right) = M\left(fx, fy, fz, \frac{t}{\lambda}\right)$$

For $t > 0$, and by Lemma 2.3 it follows that $fx = fy = fz.$ that is, $x = y = z.$

Theorem 2.6:- Let $(X, M, T_{\min})$ be a complete M-fuzzy *b*-metric space, and let $f : X \to X$. If for some $\lambda \in \left(0, \frac{1}{b^2}\right)$ such that

$$M(fx, fy, fz, t) \geq \min\left\{M\left(x, y, z, \frac{t}{\lambda}\right), M\left(fx, x, z, \frac{t}{\lambda}\right), M\left(fy, y, z, \frac{t}{\lambda}\right), M\left(fx, y, z, \frac{t}{\lambda}\right), M\left(x, fz, z, \frac{t}{\lambda}\right)\right\},$$

$$x, y, z \in X,\ t > 0, \qquad (2.11)$$

Then f has a unique fixed point in X.

Proof: Let $x_0 \in X$ and $x_{n+1} = fx_n,\ n \in ¥$. By (2.11) with $x = x_n,\ y = x_{n-1}$ and $z = x_{n-2}$, using (MF4) and the assumption that $T = T_{\min}$, we have

$$M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq \min\left\{\begin{array}{l} M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{\lambda}\right), M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{\lambda}\right), M\left(x_n, x_n, x_n, \frac{t}{\lambda}\right), \\ \min\left\{M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{b\lambda}\right), M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{b\lambda}\right)\right\}, M\left(x_n, x_n, x_n, \frac{t}{\lambda}\right) \end{array}\right\}$$

$$\geq \min\left\{M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{b\lambda}\right), M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{b\lambda}\right)\right\}, \quad n \in ¥,\ t > 0.$$

As in the proof of Theorem 2.5, by Lemma 2.3 and Corollary 2.2 it follows that

$M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{b\lambda}\right)$, and $\{x_n\}$ is a Cauchy sequence. So there exists $x \in X$ such that

$\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} M\left(x, x_n, x, t\right) = 1, \quad t > 0.$ (2.12)

Let $\sigma_1 \in (\lambda b^2, 1)$ and $\sigma_2 = 1 - \sigma_1$. By (2.11) and (b4) for $T = T_{\min}$, we have

$$M\left(fx, x, x, t\right) \geq \min\left\{M\left(fx, fx_n, fx, \frac{t\sigma_1}{b}\right), M\left(fx_n, x, x, \frac{t\sigma_2}{b}\right)\right\}$$

$$\geq \min\left\{\begin{array}{l} M\left(x, x_n, x, \frac{t\sigma_1}{b\lambda}\right), M\left(x, fx, x, \frac{t\sigma_1}{b\lambda}\right), M\left(x_n, x_{n+1}, x_{n+2}, \frac{t\sigma_1}{b\lambda}\right), \\ \min\left\{M\left(fx, x, x, \frac{t\sigma_1}{b^2\lambda}\right), M\left(x, x_n, x, \frac{t\sigma_1}{b^2\lambda}\right), M\left(x, x_{n+1}, x_{n+2}, \frac{t\sigma_1}{b^2\lambda}\right)\right\}, \\ M\left(x_{n+1}, x_n, x_{n-1} \frac{t\sigma_2}{b}\right) \end{array}\right\}$$

For all $n \in ¥$ and $t > 0,$ taking $n \to \infty$ and using (2.12), we get

$$M\left(fx, x, x, t\right) \geq \min\left\{\min\left\{1, M\left(x, fx, fx, \frac{t\sigma_1}{b\lambda}\right), 1, \min\left\{M\left(fx_n, x, x, \frac{t\sigma_1}{b^2\lambda}\right), 1\right\}, 1\right\}, 1\right\}$$

$$= M\left(fx, x, x, \frac{t\sigma_1}{b^2\lambda}\right), \qquad t > 0,$$

And by Lemma 2.3 with $\nu = \frac{b^2\lambda}{\sigma_1} \in (0,1)$ it follows that $fx = x$ and by condition (2.11), for three fixed point $x = fx, \ y = fy$ and $z = fz,$ we have

$$M\left(fx, fy, fz, t\right) \geq \min\left\{\begin{array}{l} M\left(x, y, z, \frac{t}{\lambda}\right), M\left(fx, x, z, \frac{t}{\lambda}\right), M\left(fy, y, z, \frac{t}{\lambda}\right), \\ \min\left\{M\left(fx, x, z, \frac{t}{b\lambda}\right), M\left(x, y, z, \frac{t}{b\lambda}\right)\right\}, \\ M\left(x, fy, z, \frac{t}{\lambda}\right), M\left(x, y, fz, \frac{t}{\lambda}\right) \end{array}\right\}$$

$$= \min\left\{M\left(x, y, z, \frac{t}{\lambda}\right), 1, \min\left\{1, M\left(x, y, z, \frac{t}{b\lambda}\right)\right\}\right\}$$

$$= M\left(x, y, z, \frac{t}{b\lambda}\right) = M\left(fx, fy, fz, \frac{t}{b\lambda}\right), \qquad t > 0,$$

And by Lemma 2.3 it follows that $x = y = z.$

Theorem 2.7:- Let (X, M, T), $T \geq T_p$, be a complete M-fuzzy *b*-metric space, and let $f : X \to X$. Suppose that there exists $\lambda \in \left(0, \frac{1}{b^2}\right)$ such that

$$M(fx, fy, fz, t) \geq \min\left\{\begin{array}{l} M\left(x, y, z, \frac{t}{\lambda}\right), M\left(fx, x, z, \frac{t}{\lambda}\right), M\left(fy, y, z, \frac{t}{\lambda}\right), \\ \sqrt{M\left(fx, y, z, \frac{2t}{\lambda}\right)}, M\left(x, fy, z, \frac{t}{\lambda}\right) \end{array}\right\}, \qquad (2.13)$$

For all $x, y, z \in X, t > 0,$ and there exists $x_0 \in X$ *and* $\nu \in (0,1)$ such that

$$\lim_{n\to\infty} T_{i=n}^{\infty} M\left(x_0, fx_0, fx_0, \frac{t}{v^i} \right) = 1, t > 0.$$

(2.14)

Then f has a unique fixed point in X.

Proof: Let $x_0 \in X$ and $x_{n+1} = fx_n,\ n \in ¥$. Taking $x = x_n, y = x_{n-1}$ and $z = x_{n-1}$ in condition (2.13), by (b4) and with $x = x_n,\ y = x_{n-1}$ and $z = x_{n-2}$, using (MF4) and $T \geq T_p$, we have

$$M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq \min \left\{ \begin{array}{l} M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{\lambda} \right), M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{\lambda} \right), M\left(x_n, x_n, x_n, \frac{t}{\lambda} \right), \\ \sqrt{\min\left\{ M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{b\lambda} \right). M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{b\lambda} \right) \right\}}, M\left(x_n, x_n, x_n, \frac{t}{\lambda} \right) \end{array} \right\}$$

$n \in ¥,\ t > 0.$ Since $M(x,y,z,t)$ is a b-non-decreasing in t and $\sqrt{ab} \geq \min\{a,b\}$, we obtain that

$$M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq \min \left\{ M\left(x_{n+1}, x_n, x_{n-1}, \frac{t}{b\lambda} \right), M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{b\lambda} \right) \right\}$$

For all $n \in ¥, t > 0.$ by Lemma 2.3 and 2.1 it follows that

$$M\left(x_{n+1}, x_n, x_{n-1}, t\right) \geq M\left(x_n, x_{n-1}, x_{n-2}, \frac{t}{b\lambda} \right), \quad n \in ¥, t > 0,$$

And $\{x_n\}$ is a Cauchy sequence. Since (X, M, T) is a complete space, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} M\left(x, x_n, x, t\right) = 1, \quad t > 0.$$

(2.15)

Let $\sigma_1 \in (\lambda b^2, 1)$ and $\sigma_2 = 1 - \sigma_1$. By (2.13) and (MF4) for $T \geq T_p$, We have

$$M\left(fx, x, x, t\right) \geq T\left(M\left(fx, fx_n, fx, \frac{t\sigma_1}{b} \right), M\left(fx_n, x, x, \frac{t\sigma_2}{b} \right) \right)$$

$$\geq T\left(\begin{array}{l} \min\left\{ \begin{array}{l} M\left(x, x_n, x, \frac{t\sigma_1}{b\lambda} \right), M\left(x, fx, x, \frac{t\sigma_2}{b\lambda} \right), M\left(x_n, x_{n+1}, x_{n+2}, \frac{t\sigma_1}{b\lambda} \right), \\ \sqrt{M\left(fx, x, x, \frac{t\sigma_1}{b^2\lambda} \right). M\left(x, x_n, x, \frac{t\sigma_1}{b^2\lambda} \right)}, M\left(x_n, x_{n+1}, x_{n+2} \frac{t\sigma_2}{b\lambda} \right) \end{array} \right\}, \\ M\left(x_{n+1}, x_n, x_{n-1} \frac{t\sigma_2}{b} \right) \end{array} \right).$$

$$\geq T\left(\min\left\{ \begin{array}{l} M\left(x, x_n, x_{n+1}, \frac{t\sigma_1}{b\lambda} \right), M\left(x, fx, fx, \frac{t\sigma_2}{b\lambda} \right), M\left(x_n, x_{n+1}, x_{n+2}, \frac{t\sigma_1}{b\lambda} \right), \\ \min\left\{ M\left(fx, x, x, \frac{t\sigma_1}{b^2\lambda} \right). M\left(x, x_n, x, \frac{t\sigma_1}{b^2\lambda} \right) \right\}, M\left(x_n, x_{n+1}, x_{n+2} \frac{t\sigma_2}{b\lambda} \right), \\ M\left(x_{n+1}, x_n, x_{n-1} \frac{t\sigma_2}{b} \right) \end{array} \right\} \right)$$

For all $n \in ¥$ and $t > 0,$ taking $n \to \infty$ and using (2.15), we get

$$M\left(fx,x,x,t\right)\geq T\left\{\min\left\{1,M\left(x,fx,fx,\frac{t\sigma_1}{b\lambda}\right),1,\min\left\{M\left(fx_n,x,x,\frac{t\sigma_1}{b^2\lambda}\right),1\right\},1\right\},1\right\}$$

$$=M\left(fx,x,x,\frac{t\sigma_1}{b^2\lambda}\right),\qquad t>0,$$

And by Lemma 2.3 with $\nu=\frac{b^2\lambda}{\sigma_1}\in(0,1)$ it follows that $fx=x.$

Suppose that x,y and z are fixed point for $f\cdot$ By condition (2.13) we have

$$M\left(fx,fy,fz,t\right)\geq T\left\{\begin{matrix}M\left(x,y,z,\frac{t}{\lambda}\right),M\left(fx,x,z,\frac{t}{\lambda}\right),M\left(fy,y,z,\frac{t}{\lambda}\right),\\ \sqrt{M\left(fx,x,z,\frac{t}{b\lambda}\right).M\left(x,y,z,\frac{t}{b\lambda}\right)},M\left(x,fy,z,\frac{t}{\lambda}\right),M\left(x,y,fz,\frac{t}{\lambda}\right)\end{matrix}\right\}$$

$$\geq T\left\{M\left(x,y,z,\frac{t}{\lambda}\right),1,\min\left\{1,M\left(x,y,z,\frac{t}{b\lambda}\right)\right\}\right\}$$

$$=M\left(x,y,z,\frac{t}{b\lambda}\right)=M\left(fx,fy,fz,\frac{t}{b\lambda}\right),\qquad t>0,$$

And thus by Lemma 2.3 it follows that $x=y=z.$

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