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TRACE OF POSITIVE INTEGER POWER OF ADJACENCY MATRIX OF THIRD ORDER

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Abstract: Finding the trace of positive integer power of a matrix is an important problem in matrix theory. In Graph theory an important application of the trace of positive integer power of adjacency matrix is counting the triangle in connected graph. In this paper, we attempted to extend the formula to third order trace of positive integer power of some special Adjacency matrix of connected simple graph. The key idea of our formula is to multiply k times, where k is positive integer.

Key words or phrases: Adjacency matrix, Complete graph, Trace, Positive integer.

Introduction

Trace has many applications in fields such as Matrix theory, Network analysis, Differential equations, etc., In Adjacency matrix the trace $TrA^3/6$ is the trace where, A is the adjacency matrix[4]. For the counting of triangles in a graph, it can be drawn into an adjacency matrix and this formula can be used. Traces of power of integers are connected to Euler congruence [2]

$$\text{Tr}\,(M^{p^r}) \equiv Tr(M^{p^{r-1}})(mod\,p^r)$$

Where M is any integer matrix, and p be any prime number, r is the natural number. This condition holds for all integer matrices. In common trace can be calculated by addition of the diagonal elements i.e., a_{ii} where i=1,2,...,n. But for higher order matrices, this may be impossible. Trace can also be calculated by the use of Eigen vectors given by [1],

$$TrA^q=\sum\lambda_k^q$$

Where λk denotes the Eigen values. But this is also impossible in case of higher order matrices. In order to make the calculations easy and efficient we have developed a new formula for the trace of adjacency matrix. This formula will only depend on the order of matrix.

Preliminaries

Definition 1: Trace

Trace of a $n \times n$ matrix A=a_{ij}, is defined to be the sum of the elements on the main diagonals of A .i.e. $TrA = a_{11}+a_{22}+\ldots+a_{nn}$

Definition 2: Complete graph

A Complete graph is a simple undirected graph in which each pair of distinct vertices is connected by a unique edge. For a given number of vertices, there is a unique complete graph.

Definition 3: Undirected graph

An Undirected graph is a graph that are connected together, where all the edges are bidirectional.

Definition 4: Adjacency matrix

The Adjacency matrix is a square matrix used to represent a finite graph. It takes the form as 1's with 0's on the diagonal.

Eg., $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Theorem 1: (3)

For even positive integer n and 2 × 2 real matrix A is,

$$TrA^n=\sum_{r=0}^{\frac{n}{2}}\frac{(-1)^r}{r!}\,\text{n[n-(r+1)][n-(r+2)]…[upto r terms]}(DetA)^r(TrA)^{n-2r}$$

similarly,

Theorem 2: (3). For odd positive integer n and 2 × 2 real matrix A is,

$TrA^n = \sum_{r=0}^{\frac{n-1}{2}} \frac{(-1)^n}{r!}$ n[n-(r+1)][n-(r+2)]…[upto r terms]$(DetA)^r(TrA)^{n-2r}$

Main Results

Theorem 1: If A is an adjacency matrix of a complete simple graph with n vertices, then

$$Tr\ A^K = \sum_{r=1}^{\frac{n}{3}} s(k,r)n(n-1)^r(n-2)^{k-3r}\ (n-3)^{2k-3r}$$

For positive integers k divisible by 3.

$$Tr\ A^K = \sum_{r=1}^{\lfloor\frac{n}{3}\rfloor} s(k,r)n(n-1)^r(n-2)^{k-3r}\ (n-3)^{2k-3r}$$

For positive integers k not divisible by 3. Where, $s(k,r)$ depends on k and r, defined by

$$s(k,1) = 1, s\left(k, {}^{k}/_{3}\right) = 1$$

Proof:Consider a Adjacency symmetric matrix A=a_{ij}, $1 \le i \le n, 1 \le j \le n$ where

$$a_{ij} = \begin{cases} 1, if\ i \ne j \\ 0, if\ i = j \end{cases}$$

Now,$A^3 = a_{ij}$, $1 \le i \le n, 1 \le j \le n$ where

$$a_{ij} = \begin{cases} (n-2)(n-3), if\ i \ne j \\ (n-1) \qquad\qquad ,if\ i = j \end{cases}$$

Then,$A^3 = n(n-1)$ or

$$Tr\ A^3 = \sum_{r=1}^{1} s(3,r)n(n-1)^r(n-2)^{3-3r}\ (n-3)^{2.3-3r}$$

Again,A^4=a_{ij}, $1 \le i \le n, 1 \le j \le n$ where

$$a_{ij} = \begin{cases} (n-1) + (n-2)^2, if\ i \ne j \\ (n-1)\ (n-2) \quad ,if\ i = j \end{cases}$$

Then,$A^4 = n(n-1)(n-2)$ or

$$Tr\ A^4 = \sum_{r=1}^{1} s(4,r)n(n-1)^r(n-2)^{4-3r}\ (n-3)^{2.4-3r}$$

Again,A^5=a_{ij}, $1 \le i \le n, 1 \le j \le n$ where

$$a_{ij} = \begin{cases} (n-1) + (n-2)^2 + (n-3)^3, if\ i \ne j \\ \quad (n-1)\ (n-2)(n-3) \quad ,if\ i = j \end{cases}$$

Then,$A^5 = n(n-1)(n-2)(n-3)$ or

$$Tr\ A^5 = \sum_{r=1}^{1} s(5,r)n(n-1)^r(n-2)^{5-3r}\ (n-3)^{2.5-3r}$$

Again,A^6=a_{ij}, $1 \le i \le n, 1 \le j \le n$ where

$$a_{ij} = \begin{cases} (n-1)(n-2)(n-3) + (n-2)(n-2)^2 + (n-3)^3, if\ i \ne j \\ (n-1)(n-1) + \ (n-2)^2 + (n-3)^3 \qquad\qquad , if\ i = j \end{cases}$$

Then,$A^6 = n(n-1)^2 + n(n-1)(n-2)^2(n-3) + n(n-1)(n-2)(n-3)^2$ or

$$Tr\ A^6 = \sum_{r=1}^{2} s(6,r)n(n-1)^r(n-2)^{6-3r}\ (n-3)^{2.6-3r}$$

In similar way ,we may have,

A^7=a_{ij}, $1 \le i \le n, 1 \le j \le n$ then,

$A^7 = 3n(n-1)^2(n-2) + 2n(n-1)(n-2)^3(n-3) + n(n-1)(n-2)(n-3)^3$ or

$$Tr\ A^7 = \sum_{r=1}^{2} s(7,r)n(n-1)^r(n-2)^{7-3r}\ (n-3)^{2.7-3r}$$

And,

A^8=a_{ij}, $1 \le i \le n, 1 \le j \le n$ then,

$A^8 = n(n-1)^2(n-2) + 3n(n-1)(n-2)^3(n-3) + 2n(n-1)(n-2)(n-3)^3$ or

$$Tr\ A^8 = \sum_{r=1}^{2} s(8,r)n(n-1)^r(n-2)^{8-3r}\ (n-3)^{2.8-3r}$$

Again , A^9=a_{ij}, $1 \le i \le n, 1 \le j \le n$ then,

$A^9 = n(n-1)^3 + 3n(n-1)^2(n-2)^2(n-3)^2 + n(n-1)(n-2)^4(n-3)^3$ or

$$Tr\ A^9 = \sum_{r=1}^{3} s(9,r)n(n-1)^r(n-2)^{9-3r}\ (n-3)^{2.9-3r}$$

Continuing in this way , we get a sequence of equation which may be generalized as,

$$Tr\ A^K = \sum_{r=1}^{\frac{n}{3}} s(k,r)n(n-1)^r(n-2)^{k-3r}\ (n-3)^{2k-3r}$$

For positive integers k divisible by 3.

$$Tr\ A^K = \sum_{r=1}^{\lfloor\frac{n}{3}\rfloor} s(k,r)n(n-1)^r(n-2)^{k-3r}\ (n-3)^{2k-3r}$$

For positive integers k not divisible by 3. Where, $s(k,r)$ depends on k and r, defined by

$$s(k,1) = 1, s\left(k, {}^{k}/_{3}\right) = 1$$

Hence the proof is complete.

Example: Consider a 6×6 Adjacency matrix and find TrA^3.

Proof: Here n=6, k=3. Then by our theorem,

$$Tr\,A^3 = \sum_{r=1}^{1} s(3,r)n(n-1)^r(n-2)^{3-3r}\,(n-3)^{2.3-3r}$$

$$TrA^3 = s(3,r)(6)(6-1)(6-2)^{3-3}(6-3)^{6-3}$$

TrA^3=6×5×1×27=810

$TrA^3/6$=135

Conclusion

In this paper, we attempted to extend the trace of adjacency matrix to higher order to check whether it is applicable. As mentioned above, trace has many applications in the field of mathematics. This idea of finding trace could be extended to other types of matrices.

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