https://doi.org/10.56452/2022-20-013
ISSN: 0974-5823 Vol. 7 No. 07 October, 2022

International Journal of Mechanical Engineering

On A Certain Subclass Of Meromorphic Kummer Function Connected To Hurwitz- Lerch Zeta Function

S.S.Howal
Dr. S.S.M.Pratishtans College of Education,Ahmedpur - 413 515, Latur (M.S), India.*siddharthresearcher22@gmail.com*

S.V.Jagtap
Department of Mathematics, Shivaji College,UdgirDist - 431 512, Latur (M.S), India.

P.Thrupathi Reddy
Department of Mathematics, D.R.K.Institute of Science and Technology,Bowarampet, Hyderabad- 500 043, Telangana, India. *reddypt2@gmail.com*

ABSTRACT
In this paper, we introduce and study a new subclass of meromorphic Kummer function defined by a Hurwitz-Lerch Zeta function operator and obtain coefficient estimates, growth and distortion theorem, radius of convexity,integral transforms, convex linear combinations, convolution properties and $\delta-$neighborhoods for the class $\Sigma_p(\alpha)$.

Keywords and phrases:uniformly convex, uniformly starlike, meromorphic, coefficient estimates.

AMS Subject Classification: 30C45; 30C50.

1 INTRODUCTION

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z \in C : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $f(z)$ which are all univalent in E. A function $f \in A$ is a starlike function by the order α, $0 \le \alpha < 1$, if it satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (z \in E). \tag{1.2}$$

We denote this class with $S^*(\alpha)$.

A function $f \in A$ is a convex function by the order α, $0 \le \alpha < 1$, if it satisfy

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ (z \in E). \tag{1.3}$$

We denote this class with $K(\alpha)$.

Let T denote the class of functions analytic in E that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0,\ z \in E) \tag{1.4}$$

and let $T^*(\alpha) = T \cap S^*(\alpha)$, $C(\alpha) = T \cap K(\alpha)$. The class $T^*(\alpha)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [16] and others.

A function $f \in A$ is said to be in the class of uniformly convex functions of order γ and type β, denoted by $UCV(\beta,\gamma)$, if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \tag{1.5}$$

where $\beta \geq 0, \gamma \in [-1,1)$ and $\beta + \gamma \geq 0$ and it is said to be in the class corresponding class denoted by $SP(\beta,\gamma)$, if

$$\Re\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (1.6)$$

where $\beta \geq 0, \gamma \in [-1,1)$ and $\beta + \gamma \geq 0$. Indeed it follows from (1.5) and(1.6) that

$$f \in UCV(\gamma,\beta) \Leftrightarrow zf' \in SP(\gamma,\beta). \quad (1.7)$$

For $\beta = 0$, we get respectively, the classes $K(\gamma)$ and $S^*(\gamma)$. The function of the class $UCV(1,0) \equiv UCV$ are called uniformly convex functions were introduced and studied by Goodman with geometric interpretation in [7, 8]. The class $SP(1,0) \equiv SP$ is defined by Rønning [14]. The classes $UCV(1,\gamma) \equiv UCV(\gamma)$ and $SP(1,\gamma) \equiv SP(\gamma)$ are investigated by Rønning in [13]. For $\gamma = 0$, the classes $UCV(\beta,0) \equiv \beta{-}UCV$ and $SP(\beta,0) \equiv \beta{-}SP$ are defined respectively, by Kanas and Wisniowska in [10, 11].
Further Bharathi et al. [1] and others [19] have studied and investigated interesting properties for the classes $UCV(\beta,\gamma)$ and $SP(\beta,\gamma)$.

In this context, the term hypergeometric function, first coined by Wallis in the year 1655, also known as the hypergeometric series is in the complex plane C and the open unit disk E = $\{z \in C : |z| < 1\}$. This function was discussed by Euler first, and then systematically investigated by Gauss in 1813. It is formulated as [2]:

$${}_2F_1(\varrho, v; \omega; z) = \sum_{m=0}^{\infty} \frac{(\varrho)_m (v)_m}{(\omega)_m} \frac{z^m}{m!}, \quad (\varrho, v \in \mathbb{C}, \omega \in \mathbb{C}\backslash\{0,-1,\cdots\}, |z| < 1),$$

here $(\omega)_m$ is the Pochhammer (rising) symbol and is defined as:

$$(\omega)_m = \begin{cases} 1 & m = 0 \\ \omega(\omega+1)\cdots(\omega+m-1) & m \in \mathbb{N} = \{1,2,\cdots\} \end{cases}.$$

Subsequently, in 1837, Kummer presented the Kummer function, namely confluent hyper geometric function, as a solution of a Kummer differential equation. This function is written as [2]:

$$K(\varrho;\omega,z) = \sum_{m=0}^{\infty} \frac{(\varrho)_m}{(\omega)_m} \frac{z^k}{m!} = {}_1F_1(\varrho;\omega;z) \quad (\varrho \in \mathbb{C}, \omega \in \mathbb{C}\backslash\{0,-1,\ldots\}, |z| < 1).$$

Furthermore, the Zeta functions constitute some phenomenal special functions that appear in the study of Analytic Number Theory (ANT). There are a number of generalizations of the Zeta function, such as Euler-Riemann Zeta function, Hurwitz Zeta function, and Lerch Zeta function. The EulerRiemann Zeta function plays a pioneering role in ANT, due to its advantages in discussing the merits of prime numbers. It also has fruitful implementations in probability theory, applied statistics, and physics. Euler first formulated this function, as a function of a real variable, in the first half of the 18 th century. Then, in 1859 , Riemann utilized complex analysis to expand on Euler's definition to a complex variable. Symbolized by $S(x)$, the definition was posed as the Dirichlet series:

$$S(x) = \sum_{m=1}^{\infty} \frac{1}{m^x} \quad \text{for } \Re(x) > 1.$$

Later, the more general Zeta function, currently called Hurwitz Zeta function, was also propounded by Adolf Hurwitz in 1882, as a general formula of the Riemann Zeta function considered as [12]:

$$S(\mu, x) = \sum_{m=0}^{\infty} \frac{1}{(m+\mu)^x} \quad \text{for } \Re(x) > 1, \Re(\mu) > 1.$$

More generally, the famed Hurwitz-Lerch Zeta function $f(\mu,x,z)$ is described as [3] :

$$\phi_{\mu,x}(z) = \sum_{m=0}^{\infty} \frac{z^m}{(m+\mu)^x} \text{for } \Re(x) > 1, \Re(\mu) > 1, \quad (1.8)$$

($\mu \in C\backslash Z^-_0, x \in C$ when $|z| < 1$; $\Re(x) > 1$ when $|z| = 1$).

A generalization of (1.8) was proposed by Goyal and Laddha [9] in 1997, in the following formula:

$$\psi^{\wp}_{\mu,x}(z) = \sum_{m=0}^{\infty} \frac{(\wp)_m}{m!} \frac{z^m}{(m+\mu)^x} \text{for } \Re(x) > 1, \ \Re(\mu) > 1, \quad (1.9)$$

($\wp \in C, \mu \in C\backslash Z^-_0, x \in C$ when $|z| < 1$; $\Re(x - \wp) > 1$ when $|z| = 1$).

Along with these, there are more remarkable diverse extensions and generalizations that contributed to the rise of new classes of the Hurwitz-Lerch Zeta function in ([4, 5, 17, 18]) . In this effort, by utilizing analytic techniques, a new linear (convolution) operator of morphometric functions is investigated and introduced in terms of the generalized Hurwitz-Lerch Zeta functions and Kummer functions. Moreover, sufficient stipulations are determined and examined in order for some formulas of this new operator to achieve subordination. Therefore, these outcomes are an extension for some well known outcomes of starlikeness, convexity, and close to convexity.

https://doi.org/10.56452/2022-20-013

Let Σ represent the class of normalized meromorphic functions $f(z)$ by

$$f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m \quad (1.10)$$

that are regular in the punctured unit disk $E^*=\{z:z\in C \text{ and } 0<|z|<1\}$.

Furthermore, it indicates the classes of meromorphic starlike functions of order ξ and meromorphic convex of order ξ by $\Sigma_{S^*(\xi)}$ and $\Sigma_{m(\xi)}$, $(\xi\geq 0)$, respectively (see [17, 18]).

The convolution product of two meromorphic functions $f_\ell(z)(\ell = 1,2)$ in the following formula:

$$f_\ell(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_{m,\ell}z^m \qquad (\ell=1,2)$$

is defined by

$$(f_1*f_2)(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_{m,1}a_{m,2}z^m.$$

The meromorphic Kummer function $Ke(\varrho;\omega,z)$ is formulated aszm

$$\bar{K}(\varrho;\omega,z)=\frac{1}{z}+\sum_{m=0}^{\infty}\frac{(\varrho)_{m+1}}{(\omega)_{m+1}}\frac{z^m}{(m+1)!}, \quad (1.11)$$

$$(\wp\in\mathbb{C},\omega\in\mathbb{C}\backslash\{0,-1,\cdots\},z\in\mathbb{E}^*).$$

Corresponding to (1.11) and (1.9), based on a convolution tool, we imposed the following new convolution complex operator for $f(z)\in\Sigma$ as:

$$\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)=\widetilde{K}(\varrho;\omega,z)*\mathfrak{A}_{x,\mu}(z)*f(z)$$

$$=\frac{1}{z}+\sum_{m=1}^{\infty}\Phi_m a_m z^m, \quad (1.12)$$

where

$$\Phi_n=\Phi_n(\mu,\wp,\varrho,x,\omega)=\frac{(\varrho)_{m+1}(\wp)_{m+1}}{(\omega)_{m+1}(m+1)!(m+1)!}\left(\frac{\mu+1}{\mu+m+1}\right)^x$$

and

$$A^{x,\mu}(z)=(\mu+1)^x\left[\psi_{\mu,x}^{\wp}(z)-\frac{1}{\mu^x}+\frac{1}{z(\mu+1)^x}\right]$$

$$=\frac{1}{z}+\sum_{m=1}^{\infty}\frac{(\wp)_{m+1}}{(m+1)!}\left(\frac{\mu+1}{\mu+m+1}\right)^x z^m \quad (z\in\mathbb{E}^*).$$

Now, we define a new subclass $\Sigma_p(\alpha)$ of Σ_p.

Definition 1.1. For $-1\leq\alpha<1$, we let $\Sigma_p(\alpha)$ be the subclass of Σ_p consisting of the form (1.10) and satisfying the analytic criterion

$$-\Re\left\{\frac{z(\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z))'}{\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)}+\alpha\right\}>\left|\frac{z(\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z))'}{\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)}+1\right|, \quad (1.13)$$

$\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)$ is given by (1.12).

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$neighbourhoods for the class $\Sigma_p(\alpha)$.

2 COEFFICIENT INEQUALITY

In this section we obtain the coefficient bounds of function $f(z)$ for the class $\Sigma_p(\alpha)$.

Theorem 2.1. *A function $f(z)$ of the form* (1.10) *is in $\Sigma_p(\alpha)$ if*

$$\sum_{m=1}^{\infty}\Phi_m[2m+3-\alpha]\,|a_m|\;\leq\;(1-\alpha)\;(-1\leq\alpha<1). \quad (2.1)$$

Proof. It sufficient to show that

$$\left|\frac{z(\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z))'}{\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)}+1\right|+\Re\left\{\frac{z(\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z))'}{\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)}+1\right\}\leq(1-\alpha).$$

We have

$$\left|\frac{z(\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z))'}{\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)}+1\right|+\Re\left\{\frac{z(\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z))'}{\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)}+1\right\}$$

$$\leq 2\left|\frac{z(\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z))'}{\mathscr{L}_\mu^x(\varrho,\omega,\wp)f(z)}+1\right|$$

$$\leq\frac{2\sum_{m=1}^{\infty}\Phi_m(m+1)|a_m||z^m|}{\frac{1}{|z|}-\sum_{m=1}^{\infty}\Phi_m|a_m||z^m|}.$$

Letting $z\to 1$ alongthe real axis, we obtain

$$\leq\frac{2\sum_{m=1}^{\infty}\Phi_m(m+1)|a_m|}{1-\sum_{m=1}^{\infty}\Phi_m|a_m|}.$$

The above expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}\Phi_m[2m+3-\alpha]\,|a_m|\;\leq\;(1-\alpha).$$

Hence the theorem is completed.□

Corollary 2.2. *Let the function $f(z)$ defined by* (1.10) *be in the class $\Sigma_p(\alpha)$. Then*

$$a_m\leq\frac{(1-\alpha)}{\sum_{m=1}^{\infty}\Phi_m[2m+3-\alpha]} \quad (m\geq 1). \quad (2.2)$$

Equality holds for the function of the form

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{\Phi_m[2m+3-\alpha]}z^m. \quad (2.3)$$

https://doi.org/10.56452/2022-20-013

3 DISTORTION THEOREMS

In this section we obtain Distortion bounds for the class $\Sigma_p(\alpha)$.

Theorem 3.1. *Let the function f(z) defined by* (1.10) *be in the class* $\Sigma_p(\alpha)$. *Then for* $0<|z|=r<1$, (3.1)

$$\frac{1}{r}-\frac{(1-\alpha)}{\Phi_1(5-\alpha)}r\le|f(z)|\le\frac{1}{r}+\frac{(1-\alpha)}{\Phi_1(5-\alpha)}r$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)}{\Phi_1(5-\alpha)}z \text{ at } z=r,ir. \quad (3.2)$$

Proof. Suppose $f(z)$ is in $\Sigma_p(\alpha)$. In view of Theorem 2.1, we have

$$\Phi_1(5-\alpha)\sum_{m=1}^{\infty}a_m$$

$$\le\sum_{m=1}^{\infty}\Phi_m[2m+3-\alpha]$$

$$\le(1-\alpha)$$

which evidently yields $\sum_{m=1}^{\infty}a_m\le\frac{1-\alpha}{\Phi_1(5-\alpha)}$.

Consequently, we obtain

$$|f(z)|=\left|\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m\right|\le\left|\frac{1}{z}\right|+\sum_{m=1}^{\infty}a_m|z|^m$$

$$\le\frac{1}{r}+r\sum_{m=1}^{\infty}a_m$$

$$\le\frac{1}{r}+\frac{1-\alpha}{\Phi_1(5-\alpha)}r.$$

Also,

$$|f(z)|=\left|\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m\right|\ge\left|\frac{1}{z}\right|-\sum_{m=1}^{\infty}a_m|z|^m$$

$$\ge\frac{1}{r}-r\sum_{m=1}^{\infty}a_m$$

$$\ge\frac{1}{r}-\frac{1-\alpha}{\Phi_1(5-\alpha)}r.$$

Hence the results (3.1) follow.□

Theorem 3.2. *Let the function f(z) defined by* (1.10) *be in the class* $\Sigma_p(\alpha)$. *Then for.*

$$\frac{1}{r^2}-\frac{1-\alpha}{\Phi_1(5-\alpha)}\le|f'(z)|\le\frac{1}{r^2}+\frac{1-\alpha}{\Phi_1(5-\alpha)}.$$

The result is sharp, the extremal function being of the form (2.3)

Proof. From Theorem 2.1, we have

$$\Phi_1(5-\alpha)\sum_{m=1}^{\infty}ma_m$$

$$\le\sum_{m=1}^{\infty}\Phi_m[2m+3-\alpha]$$

$$\le(1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty}ma_m\le\frac{1-\alpha}{\Phi_1(5-\alpha)}.$$

Consequently, we obtain

$$|f'(z)|\le\left|\frac{1}{r^2}+\sum_{m=1}^{\infty}ma_mr^{m-1}\right|$$

$$\le\frac{1}{r^2}+\sum_{m=1}^{\infty}ma_m$$

$$\le\frac{1}{r^2}+\frac{(1-\alpha)}{\Phi_1(5-\alpha)}.$$

Also,

$$|f'(z)|\ge\left|\frac{1}{r^2}-\sum_{m=1}^{\infty}ma_mr^{m-1}\right|$$

$$\ge\frac{1}{r^2}-\sum_{m=1}^{\infty}ma_m$$

$$\ge\frac{1}{r^2}+\frac{(1-\alpha)}{\Phi_1(5-\alpha)}.$$

This completes the proof.□

4 CLASS PRESERVING INTEGRAL OPERATORS

In this section we consider the class preserving integral operator of the form (1.10).

Theorem 4.1. *Let the function f(z) defined by* (1.10) *be in the class* $\Sigma_p(\alpha)$. *Then*

$$f(z)=cz^{-c-1}\int_0^z t^cf(t)dt=\frac{1}{z}+\sum_{m=1}^{\infty}\frac{c}{c+m+1}a_mz^m \quad (c>0) \quad (4.1)$$

is in $\Sigma_p(\delta)$, *where*

$$\delta(\alpha,c)=\frac{2(m+1)+(1-\alpha)}{2(c+m+1)+(1-\alpha)}. \quad (4.2)$$

The result is sharp for $f(z)=\frac{1}{z}+\frac{(1-\alpha)}{\Phi_1(5-\alpha)}z.$

Proof. Suppose $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m$ is in $\Sigma_p(\alpha)$. Wehave

$$f(z)=cz^{-c-1}\int_0^z t^cf(t)dt=\frac{1}{z}+\sum_{m=1}^{\infty}\frac{c}{c+m+1}a_mz^m \quad (c>0).$$

It is sufficient to show that

$$\sum_{m=1}^{\infty}\frac{(m+2)^n[2m+3-\delta]}{1-\delta}\frac{c}{c+m+1}a_m\le1. \quad (4.3)$$

https://doi.org/10.56452/2022-20-013

Since $f(z)$ is in $\Sigma_p(\alpha,\beta)$, we have

$$\sum_{m=1}^{\infty}\frac{(m+2)^n[2m+3-\alpha]}{1-\alpha}|a_m|\le 1 \qquad (4.4)$$

Thus (4.3) will be satisfied if

$$\sum_{m=1}^{\infty}\frac{[2m+3-\delta]}{1-\delta}\frac{c}{c+m+1}\le\sum_{m=1}^{\infty}\frac{[2m+3-\alpha]}{1-\alpha}.$$

Solving for δ, we obtain

$$\delta\le\frac{2m+3+-\alpha}{+(1-\alpha)}=G(m) \qquad (4.5)$$

A simple computation will show that $G(m)$ is increasing and $G(m)\ge G(1)$.
Using this, the result follows.□

5 CONVEX LINEAR COMBINATIONS AND CONVOLUTION PROPERTIES

In this section we obtain sharp for $f(z)$ is meromorphically convex of order δ and necessary and sufficient condition for $f(z)$ is in the class $\Sigma_p(\alpha,\beta)$ and also proved that convolution is in the class.

Theorem 5.1. *If the function* $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m$ *is in* $\Sigma_p(\alpha)$ *then* $f(z)$*is meromorphically convex of order* $\delta(0\le\delta<1)$ *in* $|z|<r=r(\alpha,\delta)$ *where*

$$r(\alpha,\beta,\delta)=\inf_{n\ge1}\left\{\frac{(1-\delta)\Phi_m[2m+3-\alpha]}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}.$$

The result is sharp.

Proof. Let $f(z)$ be in $\Sigma_p(\alpha)$. Then, by Theorem 2.1, we have

$$\sum_{m=1}^{\infty}\Phi_m[2m+3-\alpha]|a_m|\le(1-\alpha) \qquad (5.1)$$

It is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right|\le(1-\delta)\text{ for }|z|<r=r(\alpha,\delta),$$

Where$r(\alpha,\beta,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right|=\left|\frac{\sum_{m=1}^{\infty}m(m+1)a_m z^{m-1}}{\frac{-1}{z^2}+\sum_{m=1}^{\infty}ma_m z^{m-1}}\right|\le\frac{\sum_{m=1}^{\infty}m(m+1)a_m|z|^{m+1}}{1-\sum_{m=1}^{\infty}ma_m|z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1}\le1 \qquad (5.2)$$

By (5.1), it follows that (5.2) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1}\le\frac{\Phi_m[2m+3-\alpha]}{1-\alpha}|a_m|\quad(m\ge1)$$

Or

$$|z|\le\left\{\frac{(1-\delta)\Phi_m[2m+3-\alpha]}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \qquad (5.3)$$

Setting $|z|=r(\alpha,\beta,\delta)$ in (5.3), the result follows. The result is sharp for the function

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{\Phi_m[2m+3-\alpha]}z^m\quad(m\ge1).$$ □

Theorem 5.2. *Let* $f_0(z)=\frac{1}{z}$ *and* $f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{\Phi_m[2m+3-\alpha]}z^m\ (m\ge1)$. *Then* $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m$ *is in the class* $\Sigma_p(\alpha)$ *if and only if it can be expressed in the form* $f(z)=\omega_0 f_0(z)+\sum_{m=1}^{\infty}\omega_m f_m(z)$, *where* $\omega_0\ge0,\omega_m\ge0,m\ge1$ *and* $\omega_0+\sum_{m=1}^{\infty}\omega_m=1$.

Proof. Let $f(z)=\omega_0 f_0(z)+\sum_{m=1}^{\infty}\omega_m f_m(z)$ with $\omega_0\ge0,\omega_m\ge0,m\ge1$ and $\omega_0+\sum_{m=1}^{\infty}\omega_m=1$. Then

$$f(z)=\omega_0 f_0(z)+\sum_{m=1}^{\infty}\omega_m f_m(z)$$

$$=\frac{1}{z}+\sum_{m=1}^{\infty}\omega_m\frac{(1-\alpha)}{\Phi_m[2m+3-\alpha]}z^m.$$

Since

$$\sum_{m=1}^{\infty}\frac{[\Phi_m[2m+3-\alpha]}{(1-\alpha)}\omega_m\frac{(1-\alpha)}{\Phi_m[2m+3-\alpha]}$$

$$=\sum_{m=1}^{\infty}\omega_m=1-\omega_0\le1.$$

By Theorem 2.1, $f(z)$ is in the class $\Sigma_p(\alpha)$.
Conversely suppose that the function $f(z)$ is in the class $\Sigma_p(\alpha)$, since

$$a_m\le\frac{(1-\alpha)}{[2m+3-\alpha]\Phi_m}z^m\quad(m\ge1).$$

$$\omega_m=\sum_{m=1}^{\infty}\frac{\Phi_m[2m+3-\alpha]}{(1-\alpha)}a_m$$

and

$$\omega_0=1-\sum_{m=1}^{\infty}\omega_m.$$

It follows that $f(z)=\omega_0 f_0(z)+\sum_{m=1}^{\infty}\omega_m f_m(z)$.

This completes the proof of the theorem.□

For the functions $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m$ and $g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_m z^m$ belongs to Σ_p, we denoted by $(f*g)(z)$ the convolution of $f(z)$ and $g(z)$ and defined as

$$(f*g)(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m b_m z^m.$$

Theorem 5.3. *If the function* $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m$ *and* $g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_m z^m$ *are in the class* $\Sigma_p(\alpha)$ *then* $(f*g)(z)$ *is in the class* $\Sigma_p(\alpha)$.

Proof. Suppose $f(z)$ and $g(z)$ are in $\Sigma_p(\alpha)$. By Theorem 2.1, we have $\sum_{m=1}^{\infty}\frac{\Phi_m[2m+3-\alpha]}{(1-\alpha)}a_m\le 1$

And $\sum_{m=1}^{\infty}\frac{\Phi_m[2m+3-\alpha]}{(1-\alpha)}b_m\le 1$.

Since $f(z)$ and $g(z)$ are regular are in E, so is $(f*g)(z)$. Further more

$$\sum_{m=1}^{\infty}\frac{[2m+3-\alpha]\Phi_m}{(1-\alpha)}a_m b_m$$
$$\le\sum_{m=1}^{\infty}\left\{\frac{\Phi_m[2m+3-\alpha]}{(1-\alpha)}\right\}^2 a_m b_m$$
$$\le\left(\sum_{m=1}^{\infty}\frac{[2m+3-\alpha]\Phi_m}{(1-\alpha)}a_m\right)$$
$$\times\left(\sum_{m=1}^{\infty}\frac{[2m+3-\alpha]\Phi_m}{(1-\alpha)}b_m\right)$$
$$\le 1.$$

Hence, by Theorem 2.1, $(f*g)(z)$ is in the class $\Sigma_p(\alpha)$.□

6 NEIGHBORHOODS FOR THE CLASS $\Sigma_P(\alpha,\gamma)$

In this section, we define the $\delta-$neighborhood of a function $f(z)$ and establish a relation between $\delta-$neighborhood and $\Sigma_p(\alpha,\gamma)$ class of a function.

Definition 6.1. A function $f\in\Sigma_p$ is said to in the class $\Sigma_p(\alpha,\gamma)$ if there exists a function $g\in\Sigma_p(\alpha)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<(1-\gamma)\quad(z\in E,\ 0\le\gamma<1)\tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruschweyh [15]. We defined the neighborhood of a function $f\in\Sigma_p$ by

$$N_\delta(f)=\left\{g\in\Sigma_p\ \middle|\ g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_m z^m\ :\ \sum_{m=1}^{\infty}m|a_m-b_m|\le\delta\right\}.\tag{6.2}$$

Theorem 6.2. *If* $g\in\Sigma_p(\alpha)$ *and*

$$\gamma=1-\frac{\delta(5-\alpha)}{4}\tag{6.3}$$

then $N_\delta(g)\subset\Sigma_p(\alpha,\gamma)$.

Proof. Let $f\in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty}m|a_m-b_m|\le\delta\tag{6.4}$$

which implies the coefficient of inequality $\sum_{m=1}^{\infty}|a_m-b_m|\le\delta\quad(m\in\mathbb{N})$.

Since $g\in\Sigma_p(\alpha)$, we have $\sum_{m=1}^{\infty}b_m=\frac{1-\alpha}{(5-\alpha)}$.

Sothat $\left|\frac{f(z)}{g(z)}-1\right|<\frac{\sum_{m=1}^{\infty}|a_m-b_m|}{1-\sum_{m=1}^{\infty}b_m}\le\frac{\delta[(5-\alpha)]}{4}=1-\gamma,$

provided γ is given by (6.3).

Hence, by Definition 6.1, $f\in\Sigma_p(\alpha,\gamma)$ for γ given by (6.3), which completes the proof of theorem.□

REFERENCES

1. R. Bharathi, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamkang J. Maths., 28(1) (1997), 17-32.
2. Cuyt, V.B. Petersen, B. Verdonk, H. Waadeland and W. B. Jones, Handbook of continued fractions for special functions; Springer Science and Business Media: Berlin, Germany, 2008.
3. Erd'elyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Tables of integral transforms; McGraw-Hill Book company: New York, NY, USA; Toronto, ON, Canada; London, UK, 1954; Volume II.
4. F. Ghanim, A study of a certain subclass of Hurwitz-Lerch zeta function related to a linear operator, Abstr. Appl. Anal., 2013 (2013), Article ID. 763756.
5. Ghanim and M. Darus, New result of analytic functions related to Hurwitz-Zeta function, SCI. World J., 2013 (2013), Article ID. 475643.
6. W. Goodman, Univalent functions and non-analytic curves, Proc. Amer. Math. Soc., (1957), 598-601.
7. W. Goodman, On uniformly convex functions, Ann. Pol. Math., 56 (1991), 87-92.
8. W. Goodman, On Uniformly starlike functions, J. of Math. Anal. and Appl., 155 (1991), 364-370 .
9. S.P. Goyal and R.K. Laddha, On the generalized Zeta function and the generalized Lambert function, Ganita Sandesh, 11 (1997), 99-108.
10. S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, Comput. Appl. Math., 105 (1999), 327-336.
11. S. Kanas and A. Wisniowska, Conic domains and starlike functions, Rev. Roum. Math. Pures Appl., 45 (2000), 647-657.

https://doi.org/10.56452/2022-20-013

12. Laurincikas and R. Garunktis, The Lerch Zeta-Function; Kluwer: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2002.
13. F.Rønning, On starlike functions associated with parabolic regions, Ann. Univ. Mariae Curie-Sklodowska- Sect. A, 45 (1991), 117-122.
14. F. Rønning, Uniformly convex functions and a corresponing class of starlike functions, Proc. Amer. Math. Soc., 118 (1993), 189-196.
15. St. Ruscheweyh, Neighbourhoods of univalent functions, Proc. Amer. Math. Soc., 81 (1981), 521-527.
16. H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 (1975), 109-116.
17. H.M. Srivastava, S. Gaboury and F. Ghanim, Certain subclasses of meromorphically univalent functions defined by a linear operator associated with the -generalized Hurwitz-Lerch zeta function, Integral Transform, Spec. Func., 26 (2015), 258-272.
18. H.M. Srivastava, S. Gaboury and F. Ghanim, Some further properties of a linear operator associated with the ρ-generalized Hurwitz-Lerch zeta function related to the class of meromorphically univalent functions, Appl. Math. Comput., 259 (2015), 1019-1029.
19. Venkateswarlu, P. Thirupathi Reddy and N. Rani, Certain subclass of meromorphically uniformly convex functions with positive coefficients, Mathematica (Cluj), 61 (84) (1) (2019), 85–97 .