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Numerical Solutions Using Galerkin-Finite Element Method

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ABSTRACT

In this an attempt has been made to solve some parabolic partial differential equations by using finite differences methods.

We consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}, \qquad (x,t)\in\Omega$$

The nonlinear partial differential equation is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modelling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc

In this, a numerical algorithm for the solution of the burger's equation based on Galerkin method employing linear finite elements is developed. The performance of this algorithm is investigated b comparing solutions to two well known problems with data available in literature. The new method produces highly accurate numerical solutions for burger's equation even for small value of viscosity coefficient. The method does, in fact, produce more accurate results then many of the other methods.

Numerical Solution of Burger's Equation by Using Galerkin Finite Element Method

Introduction

Consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial U}{\partial t}+u\frac{\partial U}{\partial x}-\nu\frac{\partial^2 U}{\partial x^2}=0 \qquad (x,t)\in\Omega\times[0,T] \tag{1.1}$$

where

$$\Omega=(0,1)\times(0,t]$$

with initial condition

$$U(x,0)=f(x) \qquad 0<x<1 \tag{1.2}$$

and boundary conditions

$$U(0,t)=g_1(t) \qquad 0\le t\le T \tag{1.3}$$

$$U(1,t)=g_2(t) \qquad 0\le t\le T \tag{1.4}$$

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where $\nu = \frac{1}{R}$ and R is the Reynolds number and f , g_1 *and* g_2 are the sufficiently smooth given functions.

The nonlinear partial differential equation (1) is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modeling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman [16] in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc.

The study of the solution of Burgers' equation has been carried out for last half Century and still it is an active area of research to develop better numerical schemes to approximate its solution. In 1965, Holf and Cole [18] propose a transformation known as Holf-Cole transformation to solve the Burgers' equation. In 1972, Benton and Platzman [19] published a number of distinct solutions to the initial value problems for the Burgers' equation in the infinite domain as well as in the finite domain. Caldwell and Smith [20] use finite difference and cubic spline finite element methods to solve Burgers' equation. Evans et al. [21] introduce the group-explicit method and Kakuda et al. [22] propose a generalized boundary element approach to solve Burgers' equation. Ali et al. [23] use a cubic B-spline finite element method based on a collocation formulation to solve Burgers' equation. Mittal et al. [24] present a numerical approximation based on one dimensional Fourier expansion with time dependent coefficients. Gardner et al. [25] apply Petrov-Galerkin method with quadratic B-spline spatial finite elements and use a least squares technique using linear space-time finite elements [26]. In [27], Ozis and Ozdes generate a sequence of approximate solutions based on variational approach which converges to the exact solution. In [28], Kutluay et al. transform the Burgers' equation to linear heat equation using Hopf-Cole transformation and then use explicit finite difference and exact explicit finite difference methods to solve the transformed linear heat equation with Neumann boundary conditions. In [29], Kutluay et al. reduce Burgers' equation to a pentadiagonal matrix system by applying the classical weighted residual method over the finite elements which is solved by a variant of Thomas algorithm together with an iteration process at each time step. Ozis et al. [30] use a finite element approach for numerical solution of Burgers' equation. Kadalbajoo et al. [31] propose a parameter uniform numerical method to solve Burgers' equation with small coefficient of viscosity and establish robust error estimate. Kadalbajoo et al. [32] use Crank-Nicolson finite difference method on the transformed linear heat equation with Neumann boundary conditions and the method is proved to be unconditionally stable. Recently, Kannan and Wang [33] have developed a high order spectral volume method using the Hopf–Cole transformation for the numerical solution of Burgers' equation while Altiparmak and Özis [34] used factorized diagonal Padé approximation method for the numerical solution of Burgers' equation while Korkmaz and Dağ [9] proposed a numerical method for nonlinear Burgers' equation.

Recently, Korkmaz and Dag [15-22] proposed sinc differential quadrature method, B-spline differential quadrature methods and cosine expansion based differential quadrature method for many nonlinear partial differential equations. Mittal have used polynomial based differential quadrature method for numerical solutions of some two dimensional nonlinear partial differential equations.

In this chapter, Galerkin-finite element method is proposed for the numerical solution of Burgers' equation. A linear recurrence relationship is found for the numerical solution of resulting system of ordinary differential equations is found vai a Crank-Nocolson approach involving a product approximation. The results show that the proposed method is more accurate.

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Galerkin-Finite Element Method for Numerical Solutions of Burgers' Equation
The burger's equation

$$\frac{\partial U}{\partial t} - U\frac{\partial U}{\partial x} - v\frac{\partial^2 U}{\partial x^2} = 0 \qquad (1.5)$$

When applying Galerkin's method we minimise the functional

$$\int_{x_0}^{x_N}\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} - v\frac{\partial^2 U}{\partial x^2}\right)\phi_i dx = 0 \qquad (1.6)$$

where ϕ_i is the weight function, with respect to nodal variables.

A numerical solutions to the partial differential equation is sought over the region $x_0 \le x \le x_N$ with boundary conditions specified at $x = x_0$, $x = x_N$. the region $[x_0, x_N]$ is splitter up into uniformly sized intervals by x_i such that $x_0 < x_1 < \ldots < x_N$. A typical finite element of size $h = (x_{m-1} - x_m)$, mapped by, local coordinates η, *where* $x = x_m + \eta h,\ 0 \le \eta \le 1$, makes the integral (3.6) the contribution.

$$\int_0^1\left(\frac{\partial U}{\partial t} + \frac{\hat{U}}{h}\frac{\partial U}{\partial \eta} - \frac{v}{h^2}\frac{\partial^2 U}{\partial \eta^2}\right)\phi_i\, d\eta = 0 \qquad (1.7)$$

where to simplify the integral, $U^{\wedge}$ is taken to be constant over the element. this leads to

$$\int_0^1\left(\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial \eta} - b\frac{\partial^2 U}{\partial \eta^2}\right)\phi_i\, d\eta = 0, \qquad (1.8)$$

where $b = \frac{v}{\eta^2}$ and $v = \frac{\hat{U}}{h}$

and b and v are taken as locally constant over each element. The variation of U over the element $[x_m, x_{m+1}]$ is expressed as

$$U^e = \sum_{i=1}^{2} P_i u_i \qquad (1.9)$$

where P_1, P_2 are linear spatial basis function and u_1, u_2 are the nodal parameters. With the local coordinate system η defined above the basic functions have the following expressions [18]

$$P_1 = 1 - \eta\ , \quad P_2 = \eta\ .$$

For gale kin's method we identify the weight function ϕ_i with basis function P_i giving

$$\int_0^1\left(\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial \eta} - b\frac{\partial^2 U}{\partial x^2}\right)P_i d\eta = 0 \qquad (1.10)$$

Integrating by parts leads to

$$\int_0^1\left[\left(\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial \eta}\right)P_i + b\frac{\partial U}{\partial \eta}\frac{\partial P_i}{\partial \eta}\right]d\eta = 0 \qquad (1.11)$$

Now if we substitute for U using equation (1.9) an element's contribution is found in the form

$$\int_0^1\left[P_i P_j\frac{\partial U_j}{\partial t} + vP_i\frac{\partial P_j}{\partial \eta}u_j + b\frac{\partial P_i}{\partial \eta}\frac{\partial P_j}{\partial \eta}u_j\right]d\eta = 0 \qquad (1.12)$$

In the matrix notation this becomes

$$A^e\frac{\partial u^e}{\partial t} + \left[C^e + bD^e\right]u^e = 0 \qquad (1.13)$$

Where $u^e = (u_1, u_2)^T$ are the relevant nodal parameters. The element matrices is

$$A_{ij}^e = \int_0^1 P_i P_j\, d\eta \qquad C_{ij}^e = v\int_0^1 P_i\frac{\partial P_j}{\partial \eta}d\eta \qquad D_{ij}^e = \int_0^1\frac{\partial P_i}{\partial \eta}\frac{\partial P_j}{\partial \eta}d\eta$$

And v is given as

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$v = \frac{u_1}{h}$ is constant over the element.

$$A_{ij}^{\ e} = \int_0^1 P_i P_j d\eta$$

$$(for\ i = 1 = j)$$

$$A_{11} = \int_0^1 P_1 P_1 d\eta = \int_0^1 (1-\eta)(1-\eta) d\eta = -((1-\eta)^3/3)_0^1 = \frac{1}{3}$$

$$(for\ \ i = 1, j = 2\ and\ \ j = 1, i = 2)$$

$$A_{12} = A_{21} = \int_0^1 P_1 P_2 d\eta = \int_0^1 (1-\eta)\eta\, d\eta \ = \frac{1}{6}$$

$$(for\ i = 2 = j)$$

$$A_{22}\ = \int_0^1 P_2 P_2 d\eta = \int_0^1 \eta\eta d\eta = \frac{1}{3}$$

$$A_{ij}^{\ e} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C_{ij}^{\ e} = v\int_0^1 P_i \frac{\partial P_j}{\partial \eta} d\eta$$

$for(i = 1 = j)$ $\qquad$ $for(i = 1\ \ and\ \ j = 2)$

$$C_{11} = v\int_0^1 (1-\eta)(-1)d\eta = \frac{-v}{2} \qquad C_{12} = v\int_0^1 P_1 \frac{\partial P_2}{\partial \eta} d\eta = v\int_0^1 (1-\eta)d\eta \ = -\frac{v}{2}$$

$for(\ i = 2\ ,\ j = 1)$ $\qquad$ $for(\ i = 2\ = \ j)$

$$C_{21} = v\int_0^1 \eta(-1)d\eta = \frac{-v}{2} \qquad C_{22} = v\int_0^1 \eta d\eta = \frac{v}{2}$$

$$C_{ij}^{\ e}\ = \frac{v}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D_{ij}^{\ e} = \int_0^1 \frac{\partial P_i}{\partial \eta}\frac{\partial P_j}{\partial \eta} d\eta$$

$for\ \ (i = 1 = j)$ $\qquad$ $for(i = 2 = j)$

$$D_{11} = \int_0^1 \frac{\partial P_1}{\partial \eta}\frac{\partial P_2}{\partial \eta} d\eta = \int_0^1 (-1)(-1)d\eta = 1 \qquad D_{22} = \int_0^1 \frac{\partial P_2}{\partial \eta}\frac{\partial P_2}{\partial \eta} d\eta = \int_0^1 1\times 1 d\eta = 1$$

$for(i = 1, j = 2\ and\ \ i = 2, j = 1)$

$$D_{12} = D_{21} = \int_0^1 \frac{\partial P_1}{\partial \eta}\frac{\partial P_2}{\partial \eta} d\eta \ = \int_0^1 (-1)(1)d\eta \ = -1$$

So,

$$D_{ij}^{\ e} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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By assembling together contributions from all elements we find the matrix equation

$$A\frac{\partial u}{\partial t}+\left[C+bD\right]u=0 \tag{1.14}$$

And $u=(u_0,u_1,......,u_N)^T$, contains all parameters, a typical member of the equation (1.14) is

For 3 elements $(u_{m-1}\ ,u_m,\ u_{m+1}\ )$, we have

$$\frac{1}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 2+2 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_{m-1}\\ \dot{u}_m\\ \dot{u}_{m+1}\end{bmatrix}+\left(\frac{1}{2}v\begin{bmatrix}-1 & 1 & 0\\ -1 & 1-1 & 1\\ 0 & -1 & 1\end{bmatrix}+b\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\right)\begin{bmatrix}u_{m-1}\\ u_m\\ u_{m+1}\end{bmatrix}=0$$

$$\frac{1}{6}\left[\dot{u}_{m-1}+4\dot{u}_m+4\dot{u}_{m+1}\right]+\left[\frac{-1}{2}v_{m-1}+\frac{1}{2}v_{m-1}-\frac{1}{2}v_m+\frac{1}{2}v_m\right]+\left[-b+2b-b\right]\begin{bmatrix}u_{m-1}\\ u_m\\ u_{m+1}\end{bmatrix}=0$$

$$\frac{\partial}{\partial t}\left[\frac{1}{6}u_{m-1}+\frac{2}{3}u_m+\frac{1}{6}u_{m+1}\right]=\left(\frac{1}{2}v_{m-1}+b\right)u_{m-1}-\left[\frac{1}{2}(v_{m-1}-v_m)+2b\right]u_m-\left(\frac{1}{2}v_m-b\right)u_{m+1}$$

We can use Crank-Nicolson approach in order to find a numerical solution for this ordinary differential equation. Taking a time center as $t=\left(n+\frac{1}{2}\right)\Delta t$, We can write

$$\frac{\partial u_m}{\partial t}=\frac{1}{\Delta t}\left(u_m{}^{n+1}-u_m{}^{n}\right),$$

$$u_m=\frac{1}{2}\left(u_m{}^{n+1}+u_m{}^{n}\right)$$

Hence we find the recurrence relationship

$$\left(\frac{1}{6}-\frac{b\Delta t}{2}-\frac{\Delta t}{4}v_{m-1}\right)u_{m-1}{}^{n+1}+\left(\frac{2}{3}+b\Delta t+\frac{\Delta t}{4}\left[v_{m-1}-v_m\right]\right)u_m{}^{n+1}+\left(\frac{1}{6}-\frac{b\Delta t}{2}+\frac{\Delta t}{4}v_m\right)u_{m+1}{}^{n+1}$$
$$=\left(\frac{1}{6}+\frac{b\Delta t}{2}+\frac{\Delta t}{4}v_{m-1}\right)u_{m-1}{}^{n}+\left(\frac{2}{3}-b\Delta t-\frac{\Delta t}{4}\left[v_{m-1}-v_m\right]\right)u_m{}^{n}+\left(\frac{1}{6}+\frac{b\Delta t}{2}-\frac{\Delta t}{4}v_m\right)u_{m+1}{}^{n}$$

The boundary conditions, $U(x_0,t) = 0$ and $U(x_N,t) = 0$ demands $u_0 = 0$ and $u_N = 0$.

The above set of quasi-linear equation has matrix which is tri-diagonal in form so that a solution applying the Thomas algorithm is feasible.

Numerical Experiments

In order to demonstrate the adaptability and the accuracy of the present method, we consider some test example available in the literature. The exact solutions of these examples are also available in the literature which is obtained by Hopf-Cole transformation. The numerical solutions generated by proposed method are compared with exact solution at the different nodal points.

Example1: Consider Burger's equation (1.1) with initial condition

$$u(x,0)=\sin\pi x \quad 0<x<1 \tag{1.15}$$

and homogeneous boundary conditions

$$u(0,t)=u(1,t)=0 \qquad 0\le t\le T$$

The analytic solution to this problem can be expressed as an infinite series

$$U(x,t)=\frac{2\pi\nu\sum_{n=1}^{\infty}A_n\exp\left(-n^2\pi^2\nu t\right)n\sin(n\pi x)}{A_0+\sum_{n=1}^{\infty}A_n\cos(n\pi x)\exp\left(-n^2\pi^2\nu t\right)} \tag{1.16}$$

where

$$A_0=\int_0^1\exp\left(\frac{-1}{2\pi\nu}\left(1-\cos(\pi x)\right)\right)dx,\quad A_n=2\int_0^1\exp\left(\frac{-1}{2\pi\nu}\left(1-\cos(\pi x)\right)\right)dx \tag{1.17}$$

The numerical solutions of the Example are presented in the Tables 1-2 and Figures 1-3. Table 1 shows the comparison of numerical and exact solutions at ν = 1.0 and at different times. The Table shows as we decrease step length the numerical solutions converges to the exact solutions. Similarly, Table 2 shows the comparison of numerical and exact solutions at ν = 0.1, 0.01and at different times. The Figures 1-3 show the physical behaviour of the problem at ν and different times.

Example 2: Consider Burger's equation (1.1) with initial condition

$$u(x,0)=4x(1-x)\qquad 0<x<1 \tag{1.18}$$

and boundary condition

$$u(0,t)=0=u(1,t)\qquad 0\le t\le T \tag{1.19}$$

The exact solution of example is obtained by Half-sole transformation and given by

$$U(x,t)=\frac{2\pi\nu\sum_{n=1}^{\infty}A_n\exp\left(-n^2\pi^2\nu t\right)n\sin(n\pi x)}{A_0+\sum_{n=1}^{\infty}A_n\cos(n\pi x)\exp\left(-n^2\pi^2\nu t\right)} \tag{1.20}$$

where

$$A_0=\int_0^1\exp\left(\frac{-1}{3\nu}\left(3x^2-2x^3\right)\right)dx$$

$$A_n=\int_0^1\exp\left(\frac{-1}{3\nu}\left(3x^2-2x^3\right)\right)dx \tag{1.21}$$

The numerical solutions of the Example are presented in the Tables 3-4 and Figures 4-6. Table 3 shows the comparison of numerical and exact solutions at ν = 1.0 and *t* = 0.1 The Table shows numerical solutions are good in agreement with the exact solution. Similarly, Table 4 shows the comparison of numerical and exact solutions at ν = 0.1, 0.01and at different times. The Figures 4-6 show the physical behaviour of the problem at ν and different times.

Conclusion

A numerical algorithm for the solution of the burger's equation based on Galerkin method employing linear finite elements is developed. The performance of this algorithm is investigated b comparing solutions to two well known problems with data available in literature. The new method produces highly accurate numerical solutions for burger's equation even for small value of viscosity coefficient. The method does, in fact, produce more accurate results then many of the other methods.

Table 1: Comparison of exact and analytic solutions of Example 1 at different time and x for $\nu = 1.0$

x	t	Present Method			Exact
		$h = 0.25$	$h = 0.125$	$h = 0.0625$	
0.25	0.05	0.4159	0.4155	0.4141	0.4131
	0.10	0.2524	0.2551	0.2546	0.2536
	0.15	0.1527	0.1570	0.1572	0.1566
	0.20	0.0918	0.0963	0.0967	0.0964
0.5	0.05	0.6045	0.6098	0.6100	0.6091
	0.10	0.3649	0.3724	0.3728	0.3716
	0.15	0.2190	0.2268	0.2276	0.2268
	0.20	0.1310	0.1379	0.1389	0.1385
0.75	0.05	0.4477	0.4533	0.4530	0.4502
	0.10	0.2668	0.2739	0.2743	0.2726
	0.15	0.1581	0.1646	0.1652	0.1644
	0.20	0.0938	0.0991	0.0998	0.0994

Table 2: Comparison of the numerical solution with the exact solution Example 1 at different time and x for $\nu = 0.1,\ 0.01$.

X	T	$\nu = 0.1$		$\nu = 0.01$	
		Computed solution	Exact Solution	Computed solution	Exact Solution
0.25	0.4	0.30881	0.62540	0.34229	0.34191
	0.6	0.24069	0.24074	0.26902	0.26896
	1.0	0.16254	0.16256	0.18817	0.18819
0.5	0.4	0.56955	0.56963	0.66797	0.66071
	0.6	0.44714	0.44721	0.53211	0.52942
	1.0	0.29188	0.29192	0.37500	0.37442
0.75	0.4	0.62540	0.62544	0.93680	0.91026
	0.6	0.48715	0.48721	0.77724	0.76724
	1.0	0.28744	0.28747	0.55833	0.55605

Table 3: Comparison between exact and numerical solutions of Example 2.for $\nu = 1.0$ at t=0.1

x	Present method	Exact solution
0.1	0.11271	0.11289
0.2	0.21600	0.21625
0.3	0.30023	0.30097
0.4	0.35824	0.35886
0.5	0.38311	0.38342
0.6	0.37016	0.37066
0.7	0.31899	0.32007
0.8	0.23511	0.23537
0.9	0.12410	0.12472

Table 4: Comparison with exact and existing numerical methods of Example 2 at different times and x.

		$\nu = 0.1$		$\nu = 0.01$	
X	t	Present method	Exact solution	Present method	Exact solution
0.25	0.4	0.31748	0.31752	0.36212	0.36226
	0.6	0.24600	0.24614	0.28189	0.28204
	0.8	0.19912	0.19956	0.23001	0.23045
	1.0	0.16513	0.16560	0.19470	0.19469
	3.0	0.02734	0.02775	0.07600	0.07613
0.50	0.4	0.58414	0.58454	0.68350	0.68368
	0.6	0.45723	0.45798	0.54861	0.54832
	0.8	0.36710	0.36740	0.45323	0.45371
	1.0	0.29800	0.29834	0.38532	0.38568
	3.0	0.04045	0.04106	0.15220	0.15218
0.75	0.4	0.64562	0.64562	0.92001	0.92050
	0.6	0.50215	0.50268	0.78211	0.78299
	0.8	0.38515	0.38534	0.66223	0.66272
	1.0	0.29523	0.29586	0.56910	0.56932
	3.0	0.03021	0.03044	0.22678	0.22774

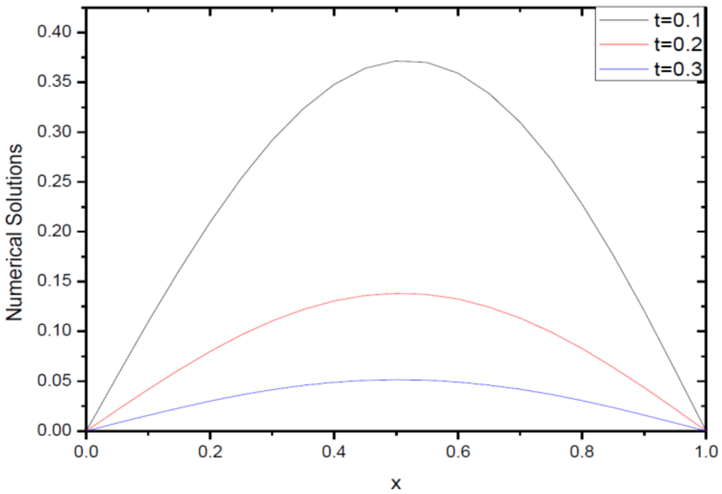

Figure 1: Numerical Solution of Example 1 at different times t and values of ν = 1.0 and Δt = 0.0001

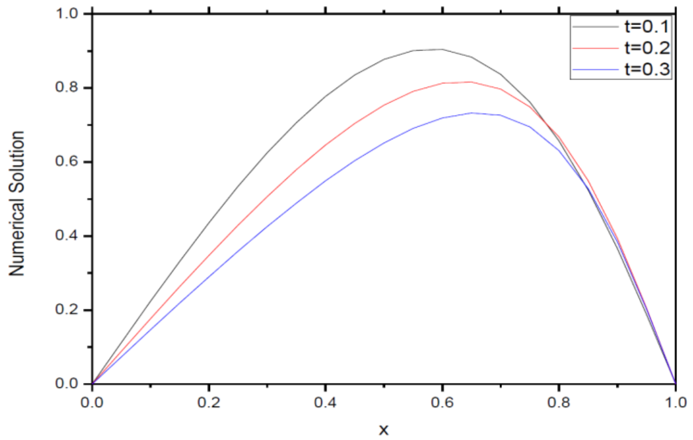

Figure 2: Numerical Solution of Example 1 at different times t and values of ν = 1.0 and Δt = 0.0001

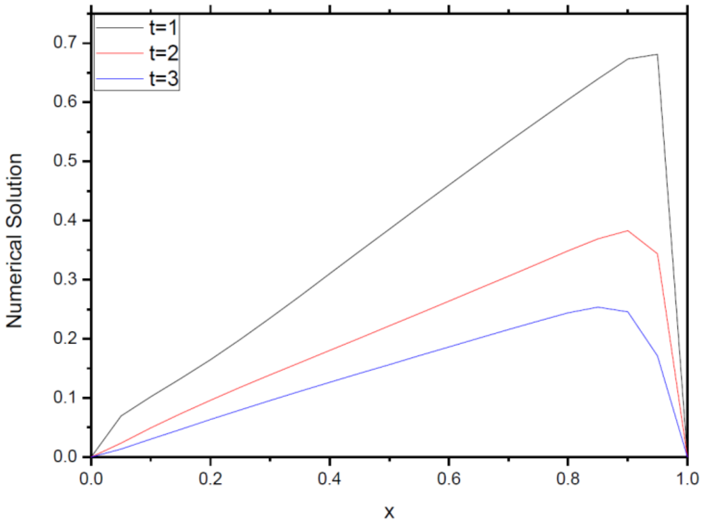

Figure 3: Numerical Solution of Example 1 at different times *t* and values of ν = 1.0 and Δt = 0.0001

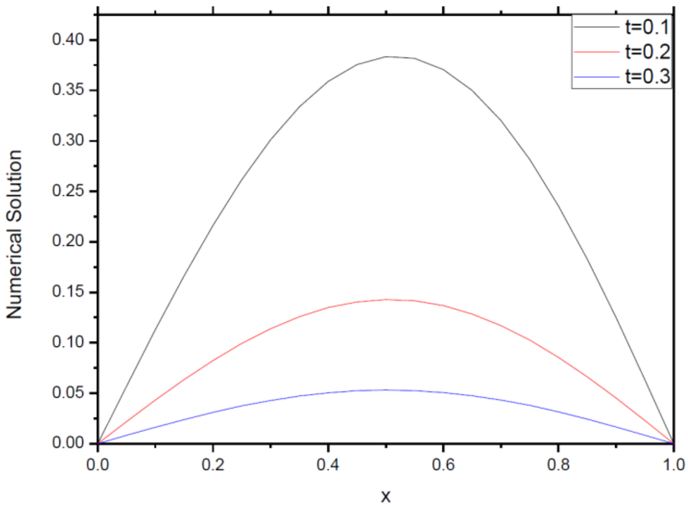

Figure 4: Numerical Solution of Example 2 at different times t and values of ν = 1.0 and Δt = 0.0001

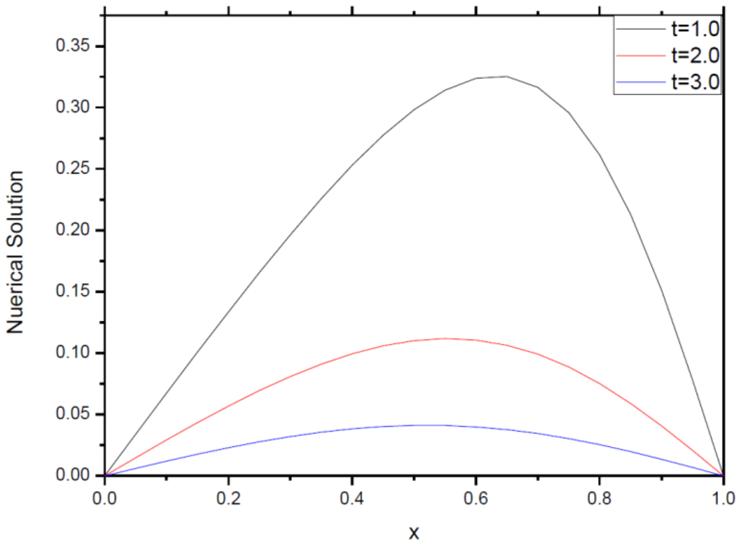

Figure 5: Numerical Solution of Example 2 at different times t and values of ν = 1.0 and Δt = 0.0001

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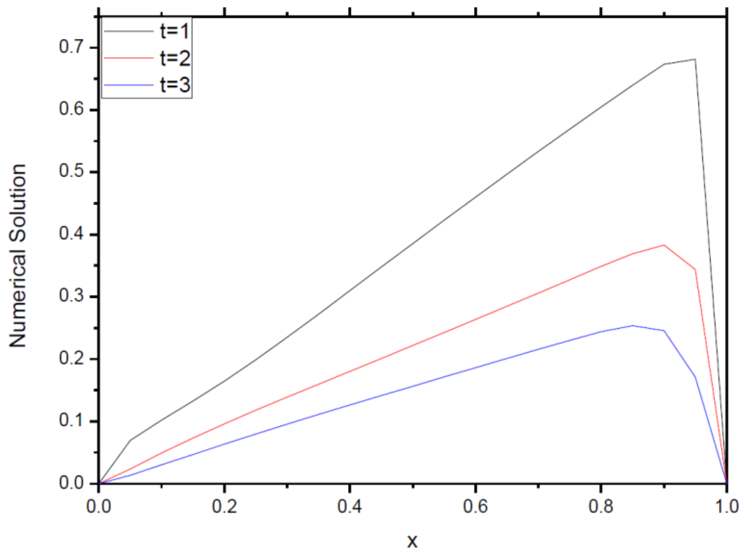

Figure 6: Numerical Solution of Example 2 at different times t and values of ν = 1.0 and Δt = 0.0001

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