ISSN: 0974-5823 Vol. 5 No. 2 December, 2020

International Journal of Mechanical Engineering

A Note on Summation of Cardinalities, Belief Functions, Plausibility Functions and Commonality Functions Induced by Probability Mass Function of Discrete Probability Distribution

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Abstract

For any subset of the discernment frame; the belief function and the plausibility function are lower and upper bounds for the probability function respectively. By replacing the probability function with its lower and upper bound, we get the lower bound and upper bound of statistical quantities such as mean, variance, raw and central moments, coefficient of skewness and coefficient of kurtosis, and so on. Now, it becomes important to study some results about the probability function's lower bound (belief function) and upper bound (plausibility function). In this paper, by using the principle of generalization, we have obtained a formula for the summation of cardinalities of all subsets of a given set with cardinality **n**. Also, for any subset of the discernment frame; we have obtained formulas for the summation of the belief functions, the commonality functions, and the plausibility functions when the probability mass function of an underlying discrete probability distribution is given.

Keywords:- cardinality of a set, Basic probability number, Belief function, Commonality function, plausibility function, probability mass function, probability distribution, etc.

1 INTRODUCTION

The number of elements in the given set is called the Cardinality of the set and is denoted by $|\cdot|$. All subsets of a given set are listed by its powerset. In [2, 9], recurrence relations are used to calculate the next term in a sequence with help of a few previous terms. Here, we provide the necessary preliminaries about the discrete belief function theory [8] and the discrete probability distribution theory [1].

1.1 The Discrete Belief Function Theory

discernment frame :

Dictionary meaning of discernment frame is a good judgment insight frame. The word discerns means to recognize or find out or hear with difficulty. If the discernment frame Θ is

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

then every element of Θ represents a proposition. A one-to-one correspondence exists between the proposition of interest and a subset of Θ. The set of all subsets of propositions of interest corresponds to the set of all subsets of Θ, that is the power set of Θ, and is denoted by 2^{Θ}. A function $m : 2^{\Theta} \to [0,1]$ is called **basic probability assignment** whenever

1. $m(\emptyset) = 0$.

2. $\sum_{A\subset\Theta} m(A) = 1$.

Here, $m(A)$ is the amount of the belief committed or assigned exactly to the subset A. The *total belief* assigned to A is a sum of $m(B)$, where B is the proper subset of A.

$$Bel(A) = \sum_{B\subset A} m(B). \tag{1}$$

If Θ is a discernment frame, then a function $Bel : 2^{\Theta} \to [0,1]$ is called a **belief function** over Θ if it satisfies above condition (1). A function $Bel : 2^{\Theta} \to [0,1]$ is *belief function* if and only if it satisfies following conditions

1. $Bel(\emptyset) = 0$.

2. $Bel(\Theta) = 1$.

3. For every positive integer n and every collection $A_1, A_2, \ldots, A_n$ of subsets of Θ

$$Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{I \subset \{1,2,\cdots,n\}} (-1)^{|I|+1} Bel(\bigcap_{i \in I} A_i). \tag{2}$$

The Degree of doubt : $Dou(A) = Bel(\bar{A})$ or $Bel(A) = 1 - Dou(\bar{A})$.

The quantity $pl(A) = 1 - Dou(A) = \sum_{A \cap B \neq \emptyset} m(B)$ expresses the amount to which one finds A credible or plausible. A relation between the belief function, the probability mass (or density) function, and the plausibility function is

$$Bel(A) \leq p(A) \leq Pl(A), \quad \forall \quad A \subset \Theta. \tag{3}$$

1.2 The Discrete Probability Distribution Theory

For discrete distribution, the probability mass function $p : \Theta \to [0, 1]$ have properties:

1 $0 \leq p(\{a\}) \leq 1, \forall \{a\} \in \Theta$.

2 $0 \leq p(A) \leq 1, \forall A \subseteq \Theta$.

3 $\sum_{\{a\} \in \Theta} p(\{a\}) = 1$.

Theorem 1.1. *A function $m : 2^{\Theta} \to [0, 1]$ is defined by*

$$m(A) = \frac{p(A)}{2^{n-1}},$$

is a basic probability assignment, where A is the subset of Θ, n is the cardinality of Θ, and $p(A)$ is the probability of A by discrete probability distribution under study [6].

If $p(\emptyset) = 0$ then $Bel(\emptyset) = 0$ and $Pl(\emptyset) = 0$ hence $Bel(\emptyset) = p(\emptyset) = Pl(\emptyset) = 0$.

In section II, using the method of generalization, a result about the summation of cardinalities of all subsets of the set having cardinality n. In section III, using this recurrence relation, a result of the summation of cardinalities of set having cardinality k+1 hence for cardinality n. In section IV, the question is raised about the set having denumerable and countably infinite cardinality. In section V, using the basic belief assignment of interest and results from the discrete belief function theory and the discrete probability distribution; a discussion about the summation of belief functions, commonality functions, and plausibility functions of all subsets of the set having cardinality up to 5 is explained hence for cardinality n, in section VI. In section VII, the result of the summation of commonality functions of all subsets of the set having cardinality n is obtained. Also, the result about the relation between the summation of belief functions and the summation of commonality functions of all subsets of the set having cardinality n. In section VIII, the result of the summation of plausibility functions of all subsets of the set having cardinality n is obtained. In section IX, the conclusion is quoted, and in section X, the relevance and applications of existing theory is explained. Finally, references are provided. The main aim is to identify patterns present in the result for the subset having cardinality k and the subset of cardinality $k+1$ containing this subset having cardinality k.

2 Recurrence Relation in Summation of Cardinalities of Subsets

Theorem 2.1. *If cardinalities of subsets A_k and A_{k+1} of discernment frame Θ are k and $k+1$ respectively, where A_{k+1} is obtained from A_K by adding one more element, then recurrence relation between the summation of cardinalities of all its subsets of A_k and A_{k+1} is $D_{k+1} = 2D_k + 2^k, k = 0, 1, 2, 3, 4, 5, \ldots$, where D_k is the sum of cardinalities of all the subsets of A_k having cardinality k.*

Proof: Here by considering some initial values in the generalization principle, we obtain a formula for the summation of cardinalities of all subsets of a given set.

Case 1: $|A| = 0$

Here A is empty set $\emptyset$ and empty set has only one subset viz. empty set $\emptyset$ itself. Thus, the summation of cardinalities of all subsets of A is 0.

Case 2: $|A| = 1$

Here $A = \{a\}$ have two subsets viz. empty set $\emptyset$ and A itself. Thus, the summation of cardinalities of all subsets of A is 1.

Case 3: $|A| = 2$

If $A = \{a, b\}$, then A has the following subsets with cardinalities as shown below:

Subset	:	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\sum \lvert\cdot\rvert$
Cardinality	:	0	1	1	2	4.

Thus, the summation of cardinalities of all subsets of A is 4.

Case 4: $|A| = 3$

If $A = \{a, b, c\}$, then A has the following subsets with cardinalities as shown below:

Subset	:$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$	$\sum \lvert\cdot\rvert$
Cardinality	:0	1	1	1	2	2	2	3	12

Thus, the summation of cardinalities of all subsets of A is 12.

Case 5 $|A| = 4$

If $A = \{a, b, c, d\}$, then A has the following subsets with cardinalities as shown below:

Subset	: $\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,c\}$
Cardinality	: 0	1	1	1	1	2	2
Subset	: $\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	
Cardinality	: 2	2	2	2	3	3	
Subset	: $\{a,c,d\}$	$\{b,c,d\}$	$\{a,b,c,d\}$	$\sum\lvert\cdot\rvert$			
Cardinality	: 3	3	4	32			

Thus, the summation of cardinalities of all subsets of A is 32.

Case 6: $|A| = 5$

If $A = \{a, b, c, d\}$ then A has the following subsets with cardinalities as shown below:

Subset	: $\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
Cardinality	: 0	1	1	1	1	1
Subset	: $\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,e\}$	$\{b,c\}$	$\{b,d\}$
Cardinality	: 2	2	2	2	2	2
Subset	: $\{b,e\}$	$\{c,d\}$	$\{c,e\}$	$\{d,e\}$	$\{a,b,c\}$	$\{a,b,d\}$
Cardinality	: 2	2	2	2	3	3
Subset	: $\{a,b,e\}$	$\{a,c,d\}$	$\{a,c,e\}$	$\{a,d,e\}$	$\{b,c,d\}$	
Cardinality	: 3	3	3	3	3	
Subset	: $\{b,c,e\}$	$\{b,d,e\}$	$\{c,d,e\}$	$\{a,b,c,d\}$	$\{a,b,c,e\}$	
Cardinality	: 3	3	3	4	4	

Subset	: $\{a,b,d,e\}$	$\{a,c,d,e\}$	$\{b,c,d,e\}$	$\{a,b,c,d,e\}$	$\sum\lvert\cdot\rvert$
Cardinality	: 4	4	4	5	80

Thus, the summation of cardinalities of all subsets of A is 80. In short, we have

$\|A\|$	:	0	1	2	3	4	5
$\sum \|\cdot\|$	:	0	1	4	12	32	80.

The pattern that occurred in the above process can be visualized as:

0

0 1

0 1 1 2

0 1 1 1 2 2 2 3

0 1 1 1 1 2 2 2 2 2 2 3 3 3 3 4

0 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 3 3 4 4 4 4 4 5

We observe that the pattern in the line above, is present in the line below and the remaining pattern also contains the same number of elements. This remaining pattern can be obtained from a pattern in the line above by adding 1 to every element. Since such pattern in the line above consists of 2^k elements and 1 is added to every element pattern in the line above to get the remaining pattern and finally, we have written both patterns combined in ascending order. Therefore we have added 2^k for obtaining the remaining pattern for the next cardinality. Here pattern for the previous cardinality (i.e. pattern in the line above) is repeated twice. In these repetitions, one pattern in the line above is kept as it is, and in the remaining pattern, we have added 1 to every element of the pattern for the previous cardinality (i.e., the pattern in the line above). Thus, the pattern for the next cardinality (pattern in the line below) is given by

$$D_{k+1} = 2D_k + 2^k, k = 0, 1, 2, 3, 4, 5, \ldots. \quad (4)$$

Where, D_k is the sum of cardinalities of all subsets of A having cardinality k. By putting values k we can verify the above relation with earlier discussion viz. $D_0 = 0, D_1 = 1, D_2 = 4, D_3 = 12, D_4 = 32, D_5 = 80$ and so on.

3 Formula for Summation of Cardinalities of Subsets

Now, we obtain explicit formula to get the summation of cardinalities of all subsets of a given set.

Theorem 3.1. *If cardinality of subset A_{k+1} of discernment frame Θ is $k+1$ then summation of cardinalities of all its subsets is $D_{k+1} = (k+1)2^k$.*

Proof: Consider recurrence relation

$$\begin{aligned}
D_{k+1} &= 2D_k + 2^k \\
&= 2(2D_{k-1} + 2^{k-1}) + 2^k \\
&= 2^2 Dk - 1 + 2(2^k) \\
&= 2^2(2Dk - 2 + 2^{k-2}) + 2(2^k) \\
&= 2^3 D_{k-2} + 2^k + 2(2^k) \\
&= 2^3 D_{k-2} + 3(2^k) \\
&= 2^3(2D_{k-3} + 2^{k-3}) + 3(2^k) \\
&= 2^4 D_{k-3} + 2^k + 3(2^k)
\end{aligned}$$

$$\begin{aligned}
D_{k+1} &= 2^4 D_{k-3} + 4(2^k) \\
&\vdots \\
&= 2^{(k-1)+1} D_{k-(k-1)} + k(2^k) \\
&= 2^k D_1 + k(2^k) \\
&= 2^k(1) + k(2^k) \\
&= (k+1)2^k.
\end{aligned}$$

Therefore, $D_{k+1} = (k+1)2^k$.

Remark: Above formula satisfies $D_0 = 0, D_1 = 1, D_2 = 4, D_3 = 12, D_4 = 32, D_5 = 80$ and so on i.e. equation (4).

Thus, we have an explicit formula to get the summation of cardinalities of all subsets of a given set of cardinality n as:

$$f(n) = n2^{n-1}, \quad n = 0, 1, 2, 3, 4, 5, \ldots. \tag{5}$$

where; n is the cardinality of a given set and $f(n)$ is the summation of cardinalities of all subsets of a given set.

4 Generalization to Further Sets

Above formula is useful and gives an appropriate answer if n is finite. If we take n as countably infinite i.e. denumerable then we have a cardinality of an infinite set as $\aleph_0$ (aleph not). We have result showing relation between $\aleph_0$ and continuum C as: $2^{\aleph_0} = C$ [3, 7]. Keeping these results in mind and applying our formula for n to the countably infinite set then we have

$$f(\aleph_0) = \aleph_0 2^{\aleph_0 - 1} = \aleph_0 2^{\aleph_0} = \aleph_0 C = C.$$

It raises a question: whether this concept can be extended to higher cardinalities viz. $2^C, 2^{2^C}$ and so on.

5 The Summation of Belief, Commonality, and Plausibility Functions

Here, we use basic probability assignment $m : 2^{\Theta} \to [0,1]$ defined by

$$m(A) = \frac{p(A)}{2^{n-1}},$$

where A is the subset of Θ, n is the cardinality of Θ, and $p(A)$ is the probability of A by discrete probability distribution under study [6]. We consider some simple cases and apply the principle of generalization.

Case 1: $|A| = 1$

Let $A = \{a\}$.

Sr. No.	*Subset of A*	$\lvert\cdot\rvert$	$m(\cdot)$	$Bel(\cdot)$	$q(\cdot)$	$Pl(\cdot)$
1	∅	0	0	0	1	0
2	{a}	1	1	1	1	1
	Total	1	1	1	2	1

Case 2 : $|A| = 2$

Let $A = \{a, b\}$.

Sr. No.	*Subset of A*	$\lvert\cdot\rvert$	$m(\cdot)$	$Bel(\cdot)$	$q(\cdot)$	$Pl(\cdot)$
1	∅	0	0	0	1	0
2	{a}	1	1/4	1/4	3/4	1
3	{b}	1	1/4	1/4	3/4	1
4	{a, b}	2	1/2	1	2/4	2
	Total	4	1	6/4	12/4	10/4

Case 3 : $|A| = 3$

Let $A = \{a, b, c\}$.

$Sr.\ No.$	$Subset\ of\ A$	$\lvert\cdot\rvert$	$m(\cdot)$	$Bel(\cdot)$	$q(\cdot)$	$Pl(\cdot)$
1	$\emptyset$	0	0	0	1	0
2	{a}	1	1/12	1/12	8/12	8/12
3	{b}	1	1/12	1/12	8/12	8/12
4	{c}	1	1/12	1/12	8/12	8/12
5	{a, b}	2	2/12	4/12	5/12	11/12
6	{a, c}	2	2/12	4/12	5/12	11/12
7	{b, c}	2	2/12	4/12	5/12	11/12
8	{a, b, c}	3	3/12	1	3/12	12/12
	Total	12	1	27/12	54/12	69/12

Case 4 : $|A| = 4$

Let $A = \{a, b, c, d\}$.

Sr. No.	*Subset of A*	$\|\cdot\|$	$m(\cdot)$	$Bel(\cdot)$	$q(\cdot)$	$Pl(\cdot)$
1	$\emptyset$	0	0	0	1	0
2	{a}	1	1/32	1/32	20/32	20/32
3	{b}	1	1/32	1/32	20/32	20/32
4	{c}	1	1/32	1/32	20/32	20/32
5	{d}	1	1/32	4/32	20/32	20/32
6	{a, b}	2	2/32	4/32	20/32	28/32
7	{a, c}	2	2/32	4/32	20/32	28/32
8	{a, d}	2	2/12	4/32	12/32	28/32
9	{b, c}	2	2/12	4/32	12/32	28/32
10	{b, d}	2	2/12	4/32	12/32	28/32
11	{c, d}	2	2/12	4/32	12/32	28/32
12	{a, b, c}	3	3/32	12/32	7/32	31/32
13	{a, b, d}	3	3/32	12/32	7/32	31/32
14	{a, c, d}	3	3/32	12/32	7/32	31/32
15	{b, c, d}	3	3/32	12/32	7/32	31/32
16	{a, b, c, d}	4	4/32	1	4/32	32/32
	Total	32	1	108/32	216/32	404/32

Now, we present the above information, in short, in the following table as:

Sr. No.	$\|\cdot\|$	$\sum\|\cdot\|$	$\sum Bel(\cdot)$	$\sum q(\cdot)$	$\sum Pl(\cdot)$
1	1	1	1	2	1
2	2	4	6/4	12/4	10/4
3	3	12	27/12	54/12	69/12
4	4	32	108/32	216/32	404/32
5	5	80	405/80	810/80	2155/80

Now, we discuss the summation of belief functions, commonality functions, and plausibility function with $m(\emptyset) = 0$ hence $Bel(\emptyset) = 0, q(\emptyset) = 1$ and $pl(\emptyset) = 0$, as follows:

6 The Summation of Belief Functions of all subsets of Θ

Theorem 6.1. *If cardinality of discernment frame Θ is n then summation of belief functions of all its subsets is $\sum_{A\subseteq\Theta} Bel(A) = (\frac{3}{2})^{n-1}$, provided basic probability assignment $m : 2^{\Theta} \to [0,1]$ is defined by $m(A) = \frac{p(A)}{2^{n-1}}$, where A is the subset of Θ and $p(A)$ is the probability of A by the discrete probability distribution under study.*

Proof: We apply principle of generalization on $|\Theta|$.

Case 1:$|\Theta| = 1$

Let $\Theta = \{a\}$.

$$\sum_{A\subseteq\Theta} Bel(A) = Bel(\emptyset) + Bel(\{a\}) = 0 + m(\{a\}) = m(\{a\} = 1 \cdot m(\{a\}.$$

Case 2:$|\Theta| = 2$

Let $\Theta = \{a, b\}$.

$$\begin{aligned}\sum_{A\subseteq\Theta} Bel(A) &= Bel(\emptyset) + Bel(\{a\}) + Bel(\{b\}) + Bel(\{a,b\}) \\ &= 0 + m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) + m(\{a,b\}) \\ &= m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) \\ &= 3 \cdot \{m(\{a\}) + m(\{b\})\}.\end{aligned} \tag{6}$$

Case 3:$|\Theta| = 3$

Let $\Theta = \{a, b, c\}$.

$$
\begin{aligned}
\sum_{A\subseteq\Theta} Bel(A) &= Bel(\emptyset) + Bel(\{a\}) + Bel(\{b\}) + Bel(\{c\}) \\
&+ Bel(\{a,b\}) + Bel(\{a,c\}) + Bel(\{b,c\} + Bel(\{a,b,c\}) \\
&= 0 + m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{a,b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{a,c\}) + m(\{b\}) + m(\{c\}) + m(\{b,c\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{a,b\}) \\
&+ m(\{a,c\}) + m(\{b,c\}) + m(\{a,b,c\}) \\
&= m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{a\}) + m(\{c\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&= 9 \cdot m(\{a\}) + 9 \cdot m(\{b\}) + 9 \cdot m(\{c\}) \\
&= 9 \cdot \{m(\{a\}) + m(\{b\}) + m(\{c\})\}.
\end{aligned} \tag{7}
$$

Here $\{a\}, \{b\}, \{c\}$ are repeated the same number of times therefore we concentrate on the repetition of $\{a\}$ only. We summarise all the observations in the following table as:

Sr. No.	*Cardinality of set* Θ	*Cardinality of subset of* Θ	*Repetition of* $\{a\}$ *for calculating belief of subset*	*Number of subsets containing* $\{a\}$	*Total number of repetitions of* $\{a\}$
1	$n = 1$	1	1	1	$1 = 3^0$
2	$n = 2$	1	1	1	
		2	1+1	1	$3 = 3^1$
3	$n = 3$	1	1	1	
		2	1+1	2	
		3	1+2+1	1	$9 = 3^2$
4	$n = 4$	1	1	1	
		2	1+1	3	
		3	1+2+1	3	
		4	1+3+3+1	1	$27 = 3^3$
5	$n = 5$	1	1	1	
		2	1+1	4	
		3	1+2+1	6	
		4	1+3+3+1	4	
		5	1+4+6+4+1	1	$81 = 3^4$

From the above table, we can generalize our formula as If $|\Theta| = n$ then the total number of repetitions of $\{a\}$ is 3^{n-1}. Therefore $\sum_{A \subseteq \Theta} Bel(A) = 3^{n-1} \sum_{\{a\} \in \Theta} m(\{a\})$.

Also, we have noticed that in any discrete probability distribution, the sum of probabilities of singleton sets is 1 [1] and the sum of cardinalities of Θ is $n2^{n-1}$, where $n = |\Theta|$ (5). Therefore basic probability number for singleton set is $m(\{a\}) = \dfrac{p(\{a\})}{2^{n-1}}$ [6]. Hence,

$$\sum_{\{a\} \in \Theta} m(\{a\}) = \frac{1}{2^{n-1}} \sum_{\{a\} \in \Theta} p(\{a\}) = \frac{1}{2^{n-1}} \cdot 1 = \frac{1}{2^{n-1}}.$$

$$\text{Finally, } \sum_{A\subseteq\Theta} Bel(A) = 3^{n-1} \sum_{\{a\}\in\Theta} m(\{a\}) \\ = 3^{n-1}\frac{1}{2^{n-1}} = (\frac{3}{2})^{n-1}. \tag{8}$$

7 The Summation of Commonality Functions of all subsets of Θ

Theorem 7.1. *If the cardinality of discernment frame Θ is n then the summation of commonality functions of all its subsets is $\sum_{A\subseteq\Theta} q(A) = 2(\frac{3}{2})^{n-1}$, provided basic probability assignment $m : 2^{\Theta} \to [0,1]$ is defined by $m(A) = \frac{p(A)}{2^{n-1}}$, where A is the subset of Θ and $p(A)$ is the probability of A by the discrete probability distribution under study.*

Proof: We apply principle of generalization on $|\Theta|$.

Case 1:$|\Theta| = 1$

Let $\Theta = \{a\}$.

$$\sum_{A\subseteq\Theta} q(A) = q(\emptyset) + q(\{a\}) = 1 + m(\{a\}) = 1 + 1 \cdot m(\{a\}$$

.

Case 2:$|\Theta| = 2$

Let $\Theta = \{a, b\}$.

$$\sum_{A\subseteq\Theta} q(A) = q(\emptyset) + q(\{a\}) + q(\{b\}) + q(\{a,b\})$$

$$= 1 + m(\{a\}) + m(\{a,b\}) + m(\{b\}) + m(\{a,b\}) + m(\{a,b\})$$

$$= 1 + m(\{a\}) + m(\{a\}) + m(\{b\})$$

$$+ m(\{b\}) + m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) \tag{9}$$

$$= 1 + 4 \cdot \{m(\{a\}) + m(\{b\})\}$$

$$= 1 + 4 \cdot (1/2)$$

$$= 3$$

Case 3: $|\Theta| = 3$

Let $\Theta = \{a, b, c\}$.

$$
\begin{aligned}
\sum_{A\subseteq\Theta} q(A) &= q(\emptyset) + q(\{a\}) + q(\{b\}) + q(\{c\}) \\
&+ q(\{a,b\}) + q(\{a,c\}) + q(\{b,c\} + q(\{a,b,c\}) \\
&= 1 + m(\{a\}) + m(\{a,b\}) + m(\{a,c\}) + m(\{a,b,c\}) \\
&+ m(\{b\}) + m(\{a,b\}) + m(\{b,c\}) + m(\{a,b,c\}) \\
&+ m(\{c\}) + m(\{a,c\}) + m(\{b,c\}) + m(\{a,b,c\}) \\
&+ m(\{a,b\}) + m(\{a,b,c\}) + m(\{a,c\}) + m(\{a,b,c\}) \\
&+ m(\{b,c\}) + m(\{a,b,c\}) + m(\{a,b,c\}) \\
&= 1 + m(\{a\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{b\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{c\}) + m(\{a\}) + m(\{c\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&= 1 + 14 \cdot \{m(\{a\}) + m(\{b\}) + m(\{c\})\} \\
&= 1 + 14 \cdot (1/4) \\
&= 18/4.
\end{aligned}
\tag{10}
$$

Here we have observed that every singleton set is repeated the same number of times.

Therefore WOLG, we concentrate on the repetition of a singleton set $\{a\}$. We will classify according to cases:

1 subset of Θ contains $\{a\}$ directly and

2 subset of Θ contains $\{a\}$ indirectly.

We summarise it in the following table as:

Sr. No.	*Cardinality of set Θ*	*Cardinality of subset of Θ containing $\{a\}$*	*Total no. of repetition of $\{a\}$ for calculating commonality of subset*	*No. of subsets do not containing $\{a\}$*	*Total no. repetitions of $\{a\}$ for calculating commonality of subset*	*Total no. of repetitions of $\{a\}$*	*Directly + Indirectly*
1	$n=1$	1	1	0	0	1	1+0
2	$n=2$	1	2	1	1	3	
3		2	1	0	0	1	3+1
4	$n=3$	1	4	2	4	8	
5		2	4	1	1	5	
6		3	1	0	0	1	9+5

By analogy of above procedure for $|\Theta| = 4$ i.e $\Theta = \{a, b, c, d\}$, we get

$$\sum_{A\subseteq\Theta} q(A) = 1 + 46 \cdot \{m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{d\})\}.$$

and for $|\Theta| = 5$ i.e $\Theta = \{a, b, c, d, e\}$, we get

$$\sum_{A\subseteq\Theta} q(A) = 1 + 146 \cdot \{m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{d\}) + m(\{e\})\}.$$

and for $|\Theta| = 6$ i.e $\Theta = \{a, b, c, d, e, f\}$, we get

$$\sum_{A\subseteq\Theta} q(A) = 1 + 454 \cdot \{m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{d\}) + m(\{e\}) + m(\{f\})\}.$$

By observing the above results, we get the relation:

$$\sum_{A\subseteq\Theta} q(A) = 1 + K \cdot \{ \sum_{\{a\}\in\Theta} m(\{a\})\}.$$

Now we will see the nature of K. For this purpose, we summarize some values of K as follows:

$\|\Theta\| = n:$	0	1	2	3	4	5	6	7
$K:$	0	1	4	14	46	146	454	1394.

Values of K are written as:

$$\begin{aligned}
0 &= 0 \\
1 &= \mathbf{1} + 0 \\
4 &= \mathbf{2} + \mathbf{1} + 1 \\
14 &= \mathbf{4} + \mathbf{2} + \mathbf{2} + \mathbf{1} + 2 + 1 + 1 + 1 \qquad (11) \\
46 &= \mathbf{8} + \mathbf{4} + \mathbf{4} + \mathbf{2} + \mathbf{4} + \mathbf{2} + \mathbf{2} + \mathbf{1} + 4 + 4 + 4 + 2 + 2 + 2 + 1 + 1 \\
146 &= \mathbf{16} + \mathbf{8} + \mathbf{8} + \mathbf{4} + \mathbf{8} + \mathbf{4} + \mathbf{4} + \mathbf{2} + \mathbf{8} + \mathbf{4} + \mathbf{4} + \mathbf{2} + \mathbf{4} + \mathbf{2} + \mathbf{2} + \mathbf{1} \\
&\quad + 4 + 2 + 2 + 1 + 2 + 1 + 1 + 1 + 2 + 1 + 1 + 1.
\end{aligned}$$

Here, in values of K, values obtained by subsets of Θ, containing $\{a\}$ are written in boldface numbers, and values obtained by subsets of Θ, not containing $\{a\}$ are written in usual numbers. These values of K are written for different values of n as,

$$1 = 1$$

$$4 = 3 + 1$$

$$14 = 9 + 3 + 2(1)$$

$$46 = 27 + 9 + 2(3 + 2 + 1)$$

$$146 = 81 + 27 + 2(9 + 2 \cdot 3 + 4 \cdot 1)$$

$$454 = 243 + 81 + 2(27 + 2 \cdot 9 + 4 \cdot 3 + 8 \cdot 1).$$

Therefore we have a formula, for $|\Theta| = n$,

$$P_n = 3^{n-1} + 3^{n-2} + 2(\sum_{i=0}^{n-3} 2^i 3^{n-3-i}) = 4 \cdot 3^{n-2} + 2(\sum_{i=0}^{n-3} 2^i 3^{n-3-i})$$

.

Now, we have for $|\Theta| = n$, $\sum_{A\subseteq\Theta} Bel(A) = (\frac{3}{2})^{n-1}$ and

$$\begin{aligned}
\sum_{A\subseteq\Theta} q(A) &= 1 + (4 \cdot 3^{n-2} + 2(\sum_{i=0}^{n-3} 2^i 3^{n-3-i})) \cdot \frac{n}{n2^{n-1}} \\
&= 1 + \frac{4 \cdot 3^{n-2} + 2(\sum_{i=0}^{n-3} 2^i 3^{n-3-i})}{2^{n-1}} \\
&= \frac{1}{2^{n-1}}(2^{n-1} + 4 \cdot 3^{n-2} + 2(\frac{3^{n-2} - 2^{n-2}}{3-2})) \\
&= \frac{1}{2^{n-1}}(2^{n-1} + 4 \cdot 3^{n-2} + 2 \cdot 3^{n-2} - 2 \cdot 2^{n-2}) \\
&= \frac{1}{2^{n-1}}(2^{n-1} + 3^{n-2}(4+2) - 2^{n-1}) \qquad (12) \\
&= \frac{1}{2^{n-1}}(3^{n-2}(6)) \\
&= \frac{2 \cdot 3}{2^{n-1}}(3^{n-2}) \\
&= \frac{2}{2^{n-1}}(3^{n-1}) \\
&= 2\frac{3^{n-1}}{2^{n-1}} \\
&= 2(\frac{3}{2})^{n-1}.
\end{aligned}$$

Remark: If the cardinality of discernment frame Θ is n then the relation between the summation of belief functions and summation of commonality functions of all its subsets is

$$\sum_{A\subseteq\Theta} q(A) = 2(\frac{3}{2})^{n-1} = 2\sum_{A\subseteq\Theta} Bel(A), \qquad (13)$$

provided basic probability number $m : \Theta \to [0,1]$ defined by $m(A) = \frac{p(A)}{2^{n-1}}$, where $p(A)$ is probability of A by discrete probability distribution under study.

8 The Summation of Plausibility Functions of all subsets of Θ

Theorem 8.1. *If the cardinality of discernment frame Θ is n then the summation of plausibility functions of all its subsets is $\sum_{A\subseteq\Theta} Pl(A) = 2^{n-1} + \sum_{r=1}^{n-1}\binom{n-1}{r}(1-2^{-r})$, provided basic probability assignment $m : 2^{\Theta} \to [0,1]$ is defined by $m(A) = \frac{p(A)}{2^{n-1}}$, where A is the subset of Θ and $p(A)$ is the probability of A by the discrete probability distribution under study.*

Proof: We apply the principle of generalization on $|\Theta|$.

Case 1:$|\Theta| = 1$

Let $\Theta = \{a\}$.

$$\sum_{A\subseteq\Theta} Pl(A) = Pl(\emptyset) + Pl(\{a\}) = 0 + m(\{a\}) = 1 \cdot m(\{a\}.$$

Case 2:$|\Theta| = 2$

Let $\Theta = \{a, b\}$.

$$
\begin{aligned}
\sum_{A\subseteq\Theta} Pl(A) &= Pl(\emptyset) + Pl(\{a\}) + Pl(\{b\}) + Pl(\{a,b\}) \\
&= 0 + m(\{a\}) + m(\{a,b\}) + m(\{b\}) + m(\{a,b\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{a,b\}) \\
&= m(\{a\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{b\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) \\
&= 5 \cdot \{m(\{a\}) + m(\{b\})\} \\
&= 5\{\frac{p(\{a\})}{2} + \frac{p(\{b\})}{2}\} \\
&= 5\{\frac{p(\{a\}) + p(\{b\})}{2}\} \\
&= 5 \cdot (1/2) \\
&= 5/2 = 10/4.
\end{aligned} \tag{14}
$$

Case 3: $|\Theta| = 3$

Let $\Theta = \{a, b, c\}$.

$$
\begin{aligned}
\sum_{A\subseteq\Theta} Pl(A) &= pl(\emptyset) + Pl(\{a\}) + Pl(\{b\}) + Pl(\{c\}) \\
&+ Pl(\{a,b\}) + Pl(\{a,c\}) + Pl(\{b,c\} + Pl(\{a,b,c\}) \\
&= m(\{a\}) + m(\{a,b\}) + m(\{a,c\}) + m(\{a,b,c\}) \\
&+ m(\{b\}) + m(\{a,b\}) + m(\{b,c\}) + m(\{a,b,c\}) \\
&+ m(\{c\}) + m(\{a,c\}) + m(\{b,c\}) + m(\{a,b,c\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{a,b\}) + m(\{a,b,c\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{a,c\}) + m(\{a,b,c\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{b,c\}) + m(\{a,b,c\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{c\}) + m(\{a,b\}) \\
&+ m(\{a,c\}) + m(\{b,c\}) + m(\{a,b,c\}) \\
&= m(\{a\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{b\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{c\}) + m(\{a\}) + m(\{c\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{a\}) + m(\{b\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) + m(\{c\}) \\
&+ m(\{b\}) + m(\{c\}) + m(\{a\}) + m(\{b\}) \\
&+ m(\{a\}) + m(\{c\}) + m(\{b\}) + m(\{c\})
\end{aligned}
$$

$$+m(\{a\})+m(\{b\})+m(\{c\})$$

$$+m(\{a\})+m(\{b\})+m(\{c\})+m(\{a\})+m(\{b\})$$

$$+m(\{a\})+m(\{c\})+m(\{b\})+m(\{c\})$$

$$+m(\{a\})+m(\{b\})+m(\{c\})$$

$$\sum_{A\subseteq\Theta} Pl(A) = 23\cdot\{m(\{a\})+m(\{b\})+m(\{c\})\} \tag{15}$$

$$= 23\cdot\{\frac{p(\{a\})}{4}+\frac{p(\{b\})}{4}+\frac{p(\{c\})}{4}\}$$

$$= 23\cdot\{\frac{p(\{a\})+p(\{b\})+p(\{c\})}{4}\}$$

$$= 23\cdot(1/4)$$

$$= 23/4.$$

Here, every singleton set is repeated the same number of times therefore WOLG, we concentrate on the repetition of $\{a\}$. Now we have the formula:

$$\sum_{A\subseteq\Theta} Pl(A) = K\cdot(\sum_{\{a\}\in\Theta} m(\{a\})$$

We summarise the values of K in the following table as:

Sr. No.	$\|\Theta\|$	$\sum Pl$	$\sum_{\{a\}\in\Theta} m(\{a\})$	Multiplier K
1	1	1	1	1
2	2	10/4	2/4	5
3	3	69/12	3/12	23
4	4	404/32	4/32	101

Now we inspect pattern of K as repetition of $\{a\}$ directly (i.e. presence of $\{a\}$ in subset of Θ while calculating sum of plausibilities all subset of Θ) and indirectly (i.e. absence of $\{a\}$ in subset of Θ while calculating sum of plausibilities all subset of Θ) in the following table as:

Sr. No.	$\|\Theta\|$	*Cardinality of subset of* Θ *containing* $\{a\}$	*Total no. of repetitions of* $\{a\}$ *for calculating plausibility of subset*	*No. of subsets do not containing* $\{a\}$	*Total no. repetitions of* $\{a\}$ *for calculating plausibility of subset*	*Total no. of repetitions of* $\{a\}$	*Directly + Indirectly*
1	$n=1$	1	1	0	0	1	1+0
2	$n=2$	1	2	1	1	3	
3		2	2	0	0	2	4+1
4	$n=3$	1	4	2	4	8	
5		2	8	1	3	11	
6		3	4	0	0	4	16+7

Now, we will write the pattern of K differently:

$\|\Theta\|$	Pattern of K
1	$1 = \mathbf{1}$
2	$5 = \mathbf{2} + \mathbf{2} + 1$
3	$23 = \mathbf{4} + \mathbf{4} + \mathbf{4} + \mathbf{4} + 2 + 2 + 3$
4	$101 = \mathbf{8} + \mathbf{8} + \mathbf{8} + \mathbf{8} + \mathbf{8} + \mathbf{8} + \mathbf{8} + \mathbf{8} + 4 + 4 + 4 + 6 + 6 + 6 + 7.$

In above table, we have written repetitions of $\{a\}$ by subsets of θ containing $\{a\}$ (i.e. directly) by boldface numbers and repetitions of $\{a\}$ by subsets of θ not containing $\{a\}$ (i.e. indirectly) by usual face numbers. From above observations and results from Hall's book [4] and Jolley'book [5], we have generalized formula for K as: for $|\Theta| = n$,

$$K = 2^{n-1}(2^{n-1}) + \binom{n-1}{1} \cdot 2^{n-2} + \binom{n-1}{2}(2^{n-2} + 2^{n-3})$$
$$+ \binom{n-1}{3}(2^{n-2} + 2^{n-3} + 2^{n-4}) + \binom{n-1}{4}(2^{n-2} + 2^{n-3} + 2^{n-4} + 2^{n-5})$$
$$+ \cdots + \binom{n-1}{n-1}(2^{n-2} + 2^{n-3} + \cdots + 2^{n-(n-1)} + 2^{n-n})$$
$$= 2^{n-1}(2^{n-1}) + \sum_{r=1}^{n-1} \binom{n-1}{r} \sum_{q=2}^{r+1} 2^{n-q}$$
$$= 2^{n-1}(2^{n-1}) + 2^{n-1} \cdot \sum_{r=1}^{n-1} \binom{n-1}{r} \sum_{q=2}^{r+1} 2^{1-q}$$
$$= 2^{n-1}\{2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r} \sum_{q=2}^{r+1} 2^{1-q}\}$$
$$= 2^{n-1}\{2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r}(1 - 2^{-r})\}$$

, Therefore, the formula for summation of plausibilities of all subsets of Θ is

$$\sum_{A \subseteq \Theta} Pl(A) = K \cdot (\sum_{\{a\} \in \Theta} m(\{a\})$$
$$= 2^{n-1}\{2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r} \sum_{q=2}^{r+1} 2^{1-q}\} \cdot \frac{n}{n2^{n-1}}$$
$$= 2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r} \sum_{q=2}^{r+1} 2^{1-q}$$
$$= 2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r} \frac{2^{1-(r+2)} - 2^{1-2}}{2^{-1} - 1} \qquad (\because \text{geometric series sum}) \qquad (16)$$
$$= 2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r} \frac{2^{-r-1} - 2^{-1}}{(1/2) - 1}$$
$$= 2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r} \frac{(1/2)(2^{-r} - 1}{-(1/2)}$$
$$= 2^{n-1} + \sum_{r=1}^{n-1} \binom{n-1}{r}(1 - 2^{-r})$$

9 Conclusion

In this paper, We have very important results about the summation of cardinalities of all subsets of a given set, the relation between the sum of beliefs, and the sum of commonalities of all subsets of the discernment frame. Also, we have obtained formulae for the sum of beliefs, commonalities, and plausibilities which are dependent only on the cardinality of subsets of the discernment frame.

10 Applications

For any set, the belief function and plausibility function are the lower and upper bounds of the probability function respectively. Every statistical quantity is dependent on probability function and statistical quantities represent medical parameters. For the statistical quantities consisting of the summation of probabilities of all subsets of a given set of interest, the summation of belief functions and summation of plausibility functions are helpful in obtaining the lower and upper bounds of such statistical quantities hence medical parameters representing them. In almost all phenomenons viz. pollution, growth, and decay; all subsets of the set of interest are involved hence the summation of belief functions, probability functions, and plausibility functions become important in further analysis of the statistical quantities consisting of the summation of probabilities of all subsets of a given set of interest.

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