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Williamson Nanofluid Flow over a Moving Surface on Boundary Layer by Homotopy Analysis Method

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Abstract - The effect of nanofluid parameters over a moving surface in the boundary layer is studied. The model of the fluid considered is Williamson and its governing equations are solved by a semi-analytic series solution called as Homotopy Analysis Method (HAM). The influence of Williamson parameters Le, Sc and Pr numbers are also studied and depicted through graphs. It has been observed that the Williamson parameters have a considerable effect on the heat transfer of nanofluid. Convergence of velocity and temperature is analyzed by plotting the h curve and Domb-Sykes plot.

Index Terms - Williamson nanofluid, Thermophoresis, Brownian motion, Pseudoplastic fluid, Domb-Sykes plot.

1 INTRODUCTION

All polymeric fluids are treated as Pseudoplastic fluids. Shear-thinning fluids are also referred to as Pseudo plastic fluids. Pseudoplastic fluids are one of the important fluids among non-Newtonian fluids. With increasing shear rate, the viscosity of these fluids will decrease. Depending on the stress-strain relationship the fluid models are named as Power-law, Carreau, Cross, Ellis, and Williamson models. Williamson (1929) proposed a fluid model known as the Williamson fluid model to describe the flow of pseudoplastic fluids. It is a simple model to simulate the shear thinning characteristics of non-Newtonian fluids [1].

There has been a growing appreciation for the study of industrial-oriented fluids such as polymeric melts and many other non-Newtonian fluids. These fluids does not obey linear relationship between shear stress and rate of strain. They exhibit a rare behaviour which are more valuable than Newtonian fluids in nature and technology. Therefore the study of different kinds of non-Newtonian fluid flows has many important applications in industry. In the study, we focus on the non-Newtonian fluid of the Williamson model. Non-Newtonian fluids are polymeric in nature which in turn is a type of nanofluid.

Nanofluids consists of nanoparticles (<100 nm) suspended in abase fluids such as water, oil and ethylene glycol. The significant heat transfer surface between particles and fluids are obtained. The thermal conductivity is the characteristic feature of nanofluid. The nanolayer acts as a thermal bridge between the solid nanoparticle and base fluid. In nanofluids, the viscosity of the base fluid affects the Brownian motion of nanoparticles which affects the thermal conductivity of the nanofluid. Nanofluids have major advantages in several Biomedical applications, such as cancer therapy, safer surgery by cooling, high power lasers, X-ray generators and magnetic cell separation.

Nadeem et al. [2,4,5,6] has explained the effect of Lewis number and Schmidt number on velocity profile for two-dimensional flow. They extended the peristaltic flow of chyme in the small intestine of Williamson flow for porous and non-porous surfaces. MHD nanofluid flow over a moving surface in the boundary layer region has been studied by Sushma et al. [3]. The MHD flow with heat and mass transfer of Williamson nanofluid over a heated surface with a variable thickness under the effect of an electric field is examined by Gossaye Aliy and Naikoti [7]. The MHD flow of Williamson fluid with variations in sheet thickness and thermal conductivity was investigated by Srinivas Reddy et al. [8]. MHD stagnation point flow exponential stretching sheet in stratified medium presented by Vittal et al. [9] using Keller-Box method. Unsteady Williamson fluid flow was studied by Bibi et al. [10].

Hamid et al. [11] investigated the two-layer non-Newtonian flow on a wedge. Hashim et al. [12] studied the MHD Williamson over a wedge with convection. Ibrahim et al. [13] included the effect of the electric field in the study of Williamson fluid flow over a radially stretching surface. Ambreen et al. [14] studied the effect of nanoparticles with boussinesq approximation on rotating disc. Kebede et al. [16] studied Williamson fluid with chemical reaction.Sakiadis [17] was the first one to initiate the study of stretching sheet problems.

Abualnaja et al. [26] presented two-dimensional steady flow by Williamson constitutive model past a nonlinear exponential stretching sheet theoretically and the system of nonlinear ordinary differential equations solved using Homotopy Perturbation Method (HPM). Gireesha et al. [27] studied the flow of Williamson fluid in a microchannel, considering the effect of thermal radiation, heat source, slip regime and convective boundary. Dawar et al. [28] investigated the Williamson nanofluid flow through a nonlinear stretching plate and analyzed the global influence of the Williamson parameter. Subbarayudu et al. [29] presented a report on Brownian motion and thermophoresis characteristics subject to MHD Williamson fluid model by assuming the flow as unsteady and also considered blood as Williamson fluid, over a wedge with radiation. The resulting equations are solved using RK 4th order method along with the Shooting technique.

In this paper, we are analyzing the Williamson fluid model of nanofluid in the boundary layer region. The distribution of the paper is as follows: section 2 involves Mathematical formulation of the problem, Method of solution of the problem is explained in section 3, results are discussed in section 4 and graphs and tables are included in section 5.

2 FORMULATION

The two-dimensional steady-state incompressible fluid flow of Williamson nanofluid is considered under the influence of Brownian motion and thermophoresis on the boundary layer flow over a moving surface, whose velocity is taken as $U_w = \lambda_1 U$, $\lambda_1 > 0$ is the velocity of the plate is in the positive direction and $\lambda_1 < 0$ is the velocity of the plate is in the negative direction. The flow along the y co-ordinate is measured perpendicular to the moving surface. U_w is the fluid velocity, T_w is the temperature, C_w is the nanoparticle concentration near the surface.
The fundamental governing equations of Williamson nanofluid are given by [2]

$$\nabla \cdot \vec{V} = 0, \tag{1}$$

$$\rho \frac{d\vec{V}}{dt} = \nabla \cdot \vec{\sigma} + \rho \vec{b}, \tag{2}$$

$$\rho c \left(\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) = \nabla \cdot k \nabla T + \rho_p c_p \times \left(D_B \nabla C \cdot \nabla T + D_T \frac{\nabla T \cdot \nabla T}{T_\infty} \right), \tag{3}$$

$$\frac{\partial C}{\partial T} + \vec{V} \cdot \nabla C = \nabla \cdot \left(D_B \nabla C + D_T \frac{\nabla T}{T_\infty} \right), \tag{4}$$

where $\vec{V},$ is the velocity, $\rho,$ is the density of the fluid, $\vec{\sigma},$ is the stress, $\vec{b},$ is the force per unit volume, ρc and $\rho_p c_p$ are the heat capacities of nanofluid and nanoparticles, $T,$ is the temperature, $k,$ is the thermal conductivity of nanofluid, $D_B,$ is Brownian diffusion coefficient, $D_T,$ is thermophoretic diffusion coefficient, $T_\infty,$ is the ambient fluid temperature and $C,$ is nanoparticle volumetric fraction. Cauchy stress tensor $\vec{\sigma}$ for Williamson fluid [15] is defined as follows

$$\vec{\sigma} = -pI + \vec{\epsilon}, \tag{5}$$

where p is the pressure, I is the stress invariant and $\vec{\boldsymbol{\epsilon}}$ is the extra stress tensor.
For $0 \le y \le h$, the pressure gradient is always negative, then

$$\vec{\epsilon} = \left[\mu_\infty + \frac{(\mu_0 - \mu_\infty)}{1 - \Gamma \dot{\beta}} \right] \tau_1, \tag{6}$$

where

$$\begin{aligned} \dot{\beta} &= \sqrt{\frac{1}{2} \delta}, \\ \delta &= trace(\tau_1^2), \end{aligned} \tag{7}$$

where τ_1 is the first Rivlin-Erickson tensor, μ_0 is limiting viscosity at zero, μ_∞ is limiting viscosity at infinite shear rate. For $\mu_\infty = 0$ and $\Gamma \dot{\beta} < 1$ equation (6) can be written as

$$\vec{\epsilon} = \left[\frac{\mu_0}{1 - \Gamma \dot{\beta}} \right] \tau_1, \tag{8}$$

or by using binomial expansion, we get

$$\vec{\epsilon} = \mu_0 [1 + \Gamma \dot{\beta}] \tau_1, \tag{9}$$

Substituting the equations (5) and (9) in equations (1) to (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{10}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \sqrt{2} \nu \Gamma \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2}, \tag{11}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\rho_p c_p}{\rho c} \left(D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right), \tag{12}$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2}, \tag{13}$$

where u and v are velocity components in $x\, and\ y$ direction, ν is the kinematic viscosity and α is the thermal diffusivity of nanofluid.
The corresponding boundary conditions are

$$u = \lambda_1 U, v = 0, T = T_\omega, C = C_\omega \text{ at } y = 0; \quad u \to U, T \to T_\infty, C \to C_\infty \text{ as } y \to \infty. \tag{14}$$

The following similarity transformations are used to reduce the governing equations into a system of ordinary differential equations.

$$\Psi = \sqrt{(2U\nu x)}F(\eta), \Theta(\eta) = \frac{T-T_\infty}{T_\omega - T_\infty}, \Phi(\eta) = \frac{C-C_\infty}{C_\omega - C_\infty}, \eta = y\sqrt{\frac{U}{2\nu x}}. \tag{15}$$

Stream function Ψ is defined as

$$u = \frac{\partial \Psi}{\partial y} \text{ and } v = -\frac{\partial \Psi}{\partial x}. \tag{16}$$

Using equation (15) in equations (11) - (13) we get the following system of ordinary differential equations

$$F''' + FF'' + \lambda F'' F''' = 0, \tag{17}$$

$$\Theta'' + PrF\Theta' + \frac{N_c}{Le}\Phi'\Theta' + \frac{N_c}{Le \times N_{bt}}\Theta'^2 = 0, \tag{18}$$

$$\Phi'' + Scf\Phi' + \frac{1}{N_{bt}}\Phi'' = 0, \tag{19}$$

where prime denotes the derivative with respect to η, $\lambda = \Gamma \frac{U^{\frac{3}{2}}}{\sqrt{\nu x}}$; Non Newtonian Williamson parameter, $Pr = \frac{\nu}{\alpha}$; Prandtl number, $Le = \frac{\alpha}{D_B}$; Lewis number, $Sc = \frac{\nu}{D_B}$; Schmidt number, $N_c = \frac{\rho_p c_p}{\rho c}(C_\omega - C_\infty)$; Heat capacities ratio and $N_{bt} = \frac{D_B T_\infty (C_\omega - C_\infty)}{D_T (T_\omega - T_\infty)}$; Diffusivity ratio.
The boundary conditions (14) becomes

$$F = 0, F' = \lambda_1, \Theta = 1, \Phi = 1 \ at \ \eta = 0,$$
$$F' = 1, \theta = 0, \Phi = 0 \ as \ \eta \to \infty. \tag{20}$$

3 SOLUTION METHODOLOGY

Equations (17) to (19) under the condition (20) are solved using the homotopy analysis method and the method is described below.

HOMOTOPY ANALYSIS METHOD:

This method is proposed by French Mathematician Jules Henry Poincare. Homotopy is a basic concept in topology, based on this the homotopy analysis method (HAM) was first developed by Shijun Liao [18-24]. HAM is independent of any small/large physical parameters. It provides us great freedom and flexibility to choose equation-type and solution expression of higher-order approximation equations and is a convenient way to guarantee the convergence of the series solution. He also proved some lemmas and theorems about the homotopy-derivative and deformation equations. Abbasbandy applied HAM to obtain the solution of nonlinear equations arising in the heat transfer field [25].

The coupled nonlinear equations for this problem are

$$N[F(\eta)] = F''' + FF'' + \lambda F'' F''', \tag{21}$$

$$N[\Theta(\eta)] = \Theta'' + PrF\Theta' + \frac{N_c}{Le}\Phi'\Theta' + \frac{N_c}{Le \times N_{bt}}\Theta'^2, \tag{22}$$

$$N[\Phi(\eta)] = \Phi'' + ScF\Phi' + \frac{1}{N_{bt}}\Theta''. \tag{23}$$

To apply the homotopy analysis method to the problem considered, we first select the auxiliary linear operator as follows to get initial approximations.

$$L_F(F) = \frac{\partial^3 F}{\partial \eta^3} + \frac{\partial^2 F}{\partial \eta^2}, \tag{24}$$

$$L_\Theta(\Theta) = \frac{\partial^2\Theta}{\partial\eta^2} + \frac{\partial\Theta}{\partial\eta}, \tag{25}$$

$$L_\Phi(\Phi) = \frac{\partial^2\Phi}{\partial\eta^2} + \frac{\partial\Phi}{\partial\eta}. \tag{26}$$

Solving (24) - (26) we get the initial approximation as follows.

$$F_0 = (\lambda_1 - 1) + \eta + (1 - \lambda_1)e^{-\eta}, \tag{27}$$

$$\Theta_0 = e^{-\eta}, \tag{28}$$

$$\Phi_0 = e^{-\eta}. \tag{29}$$

Then using HAM, we construct the zeroth-order deformation equations as follows [18].

$$(1-p)L[D(\eta,p) - F_0(\eta)] = hp\left[\frac{\partial^3 D}{\partial\eta^3} + D\frac{\partial^2 D}{\partial\eta^2} + \lambda\frac{\partial^2 D}{\partial\eta^2}\frac{\partial^3 D}{\partial\eta^3}\right], \tag{30}$$

$$(1-p)L[E(\eta,p) - \Theta_0(\eta)] = hp\left[\frac{\partial^2 E}{\partial\eta^2} + PrE\frac{\partial E}{\partial\eta} + \frac{N_c}{Le}\frac{\partial G}{\partial\eta}\frac{\partial E}{\partial\eta} + \frac{N_c}{N_{bt}\times Le}\left(\frac{\partial E}{\partial\eta}\right)^2\right], \tag{31}$$

$$(1-p)L[G(\eta,p) - \Phi_0(\eta)] = hp\left[\frac{\partial^2 G}{\partial\eta^2} + ScD\frac{\partial G}{\partial\eta} + \frac{1}{N_{bt}}\frac{\partial^2 E}{\partial\eta^2}\right]. \tag{32}$$

And the boundary conditions are

$$D(0,p) = 0, D_\eta(0,p) = \lambda_1, D_\eta(\infty,p) = 1, \tag{33}$$

$$E(0,p) = 1, \mathrm{E}(\infty,p) = 0, \tag{34}$$

$$G(0,p) = 1, \mathrm{G}(\infty,p) = 0. \tag{35}$$

When $p = 0$, we have from (30) - (32)

$$D(\eta,0) = F_0(\eta) \tag{36}$$

$$E(\eta,0) = \Theta_0(\eta) \tag{37}$$

$$G(\eta,0) = \Phi_0(\eta) \tag{38}$$

when $p = 1$, we have from (30) - (32)

$$D(\eta,1) = F(\eta) \tag{39}$$

$$E(\eta,1) = \Theta(\eta) \tag{40}$$

$$G(\eta,1) = \Phi(\eta) \tag{41}$$

Thus as p varies from 0 to 1, the solution varies from initial guess $F_0(\eta), \Theta_0(\eta)$ and $\Phi_0(\eta)$ to the exact solution $F(\eta), \Theta(\eta)$ and $\Phi(\eta)$ respectively.

For $D(\eta,p)$, $E(\eta,p)$ and $G(\eta,p)$ Maclaurin's series expansion are given by

$$D(\eta,p) = D(\eta,0) + \sum_{k=1}^{\infty}\frac{p^k}{k!}\frac{\partial^k D(\eta,p)}{\partial p^k}\bigg|_{p=0}, \tag{42}$$

$$E(\eta,p) = E(\eta,0) + \sum_{k=1}^{\infty}\frac{p^k}{k!}\frac{\partial^k E(\eta,p)}{\partial p^k}\bigg|_{p=0}, \tag{43}$$

$$G(\eta,p) = G(\eta,0) + \sum_{k=1}^{\infty}\frac{p^k}{k!}\frac{\partial^k G(\eta,p)}{\partial p^k}\bigg|_{p=0}. \tag{44}$$

Defining

$$F_0(\eta) = D(\eta,0) = \phi_0(\eta),$$
$$\phi_k(\eta) = \frac{1}{k!}\frac{\partial^k D(\eta,p)}{\partial p^k}\bigg|_{p=0}, \tag{45}$$

$$\Theta_0(\eta) = E(\eta,0) = \psi_0(\eta),$$
$$\psi_k(\eta) = \frac{1}{k!}\frac{\partial^k E(\eta,p)}{\partial p^k}\bigg|_{p=0}, \tag{46}$$

$$\Phi_0(\eta) = G(\eta,0) = \xi_0(\eta),$$
$$\xi_k(\eta) = \frac{1}{k!}\frac{\partial^k G(\eta,p)}{\partial p^k}\bigg|_{p=0}. \tag{47}$$

Using equations (45) - (47) equations (42) - (44) can be written as

$$D(\eta,p) = \phi_0(\eta) + \sum_{k=1}^{\infty} \phi_k(\eta)p^k, \tag{48}$$

$$E(\eta,p) = \psi_0(\eta) + \sum_{k=1}^{\infty} \psi_k(\eta)p^k, \tag{49}$$

$$G(\eta,p) = \xi_0(\eta) + \sum_{k=1}^{\infty} \xi_k(\eta)p^k. \tag{50}$$

The linear differential operator L and the non-zero auxiliary parameter h are selected such that the solution converges at $p = 1$. Hence the convergence region of the above series depends on L and h.

$$F(\eta) = \phi_0(\eta) + \sum_{k=1}^{\infty} \phi_k(\eta), \tag{51}$$

$$\Theta(\eta) = \psi_0(\eta) + \sum_{k=1}^{\infty} \psi_k(\eta), \tag{52}$$

$$\Phi(\eta) = \xi_0(\eta) + \sum_{k=1}^{\infty} \xi_k(\eta). \tag{53}$$

Here ϕ_m, ψ_m and ξ_m are unknowns to be determined. Differentiating equations (30), (31) and (32) , m times about the embedding parameter p, using Leibnitz theorem , setting $p = 0$ and dividing by $m!$, we get

$$L_F[\phi_m - \chi_m\phi_{m-1}] = hr_m(\eta), \tag{54}$$

$$L_\Theta[\psi_m - \chi_m\psi_{m-1}] = hs_m(\eta), \tag{55}$$

$$L_\Phi[\xi_m - \chi_m\xi_{m-1}] = ht_m(\eta). \tag{56}$$

$$\text{where} \quad \chi_m = \begin{cases} 0, & \text{when } m \le 1 \\ 1, & \text{when } m > 1 \end{cases} \quad \text{and} \tag{57}$$

$$r_m(\eta) = \phi'''_{m-1}(\eta) + \sum_{k=0}^{m-1} \phi_{m-1-k}(\eta)\phi''_k(\eta) + \lambda\sum_{k=0}^{m-1} \phi''_{m-1-k}(\eta)\phi'''_k(\eta), \tag{58}$$

$$s_m(\eta) = \psi''_{m-1}(\eta) + Pr\sum_{k=0}^{m-1} \phi_{m-1-k}(\eta)\psi'_k(\eta) + \frac{N_c}{Le}\sum_{k=0}^{\infty} \xi'_{m-1-k}(\eta)\psi'_k(\eta) + \frac{N_c}{N_{bt}\times Le}\sum_{k=0}^{m-1} \psi'_{m-1-k}(\eta)\psi'_k(\eta), \tag{59}$$

$$t_m(\eta) = \xi''_{m-1}(\eta) + Sc\sum_{k=0}^{m-1} \phi_{m-1-k}(\eta)\xi'_k(\eta) + \frac{1}{N_{bt}}\psi''_{m-1}(\eta), \tag{60}$$

with boundary conditions

$$\phi_m(0) = 0, \phi_m'(0) = 0, \phi_m'(\infty) = 0, \tag{61}$$

$$\psi_m(0) = 0, \psi_m(\infty) = 0, \tag{62}$$

$$\xi_m(0) = 0, \xi_m(\infty) = 0. \tag{63}$$

The HAM series solution for (11) is given by

$$F = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \ldots \tag{64}$$

Using MATHEMATICA, we solve (54) to get $\phi_0, \ \phi_1, \ \phi_2, \ldots$ where

$$\phi_0 = (\lambda_1 - 1) + \eta + (1 - \lambda_1)e^{-\eta}, \tag{65}$$

$$\phi_1 = \frac{1}{4}(-h + h\lambda - 2h\lambda_1 - 2h\lambda\lambda_1 + 3h{\lambda_1}^2 + h\lambda{\lambda_1}^2) + \frac{1}{4}e^{-2\eta}\Big(2e^{\eta}\big(3h - h\lambda - 6h\lambda_1 + 2h\lambda\lambda_1 + 3h{\lambda_1}^2 - h\lambda{\lambda_1}^2\big) - h(-1 + \lambda_1)\big(-1 + \lambda + \lambda_1 - \lambda\lambda_1 + 2e^{\eta}(-2 + \eta^2 + 4\lambda_1 + 2\eta\lambda_1)\big)\Big), \tag{66}$$

$$\phi_2 = \frac{1}{72}(-18h - 179h^2 + 18h\lambda + 29h^2\lambda + 6h^2\lambda^2 - 36h\lambda_1 + 6h^2\lambda_1 - 36h\lambda\lambda_1 - 42h^2\lambda\lambda_1 - 18h^2\lambda^2\lambda_1 + 54h{\lambda_1}^2 + 93h^2{\lambda_1}^2 + 18h\lambda{\lambda_1}^2 - 3h^2\lambda{\lambda_1}^2 + 18h^2\lambda^2{\lambda_1}^2 + 80h^2{\lambda_1}^3 + 16h^2\lambda{\lambda_1}^3 - 6h^2\lambda^2{\lambda_1}^3) + \frac{1}{72}e^{-3\eta}(9e^{\eta}h(2(-1 + \lambda) + h(-9 + 2\eta^2(-1 + \lambda) + \lambda(5 - 4\lambda_1) + \eta(-2 + 4\lambda(-1 + \lambda_1) - 4\lambda_1) + 2\lambda^2(-1 + \lambda_1) - 4\lambda_1))(-1 + \lambda_1)^2 - h^2(5 - 17\lambda + 12\lambda^2)(-1 + \lambda_1)^3 - 9e^{2\eta}(4h(-1 + \lambda_1)(-2 + \eta^2 + 4\lambda_1 + 2\eta\lambda_1) + \frac{1}{3}(-36h - 85h^2 + 12h\lambda + 43h^2\lambda + 72h\lambda_1 + 171h^2\lambda_1 - 24h\lambda\lambda_1 - 123h^2\lambda\lambda_1 - 36h{\lambda_1}^2 - 63h^2{\lambda_1}^2 + 12h\lambda{\lambda_1}^2 + 117h^2\lambda{\lambda_1}^2 - 23h^2{\lambda_1}^3 - 37h^2\lambda{\lambda_1}^3) + h^2(-1 + \lambda_1)(\eta^4 + 4\eta^3\lambda_1 + 2\eta^2(7 + \lambda(-1 + \lambda_1) + \lambda_1 + 2{\lambda_1}^2) + 2\eta(7 + \lambda + 12\lambda_1 - 4\lambda\lambda_1 + 5{\lambda_1}^2 + 3\lambda{\lambda_1}^2) + 4(\lambda_1(11 + 3\lambda_1) + \lambda(2 - 5\lambda_1 + 3{\lambda_1}^2))))), \tag{67}$$

etc.

The HAM series solution for (12) is given by

$$\Theta = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \ldots \tag{68}$$

Using Mathematica, we solve (55) to get $\psi_0, \ \psi_1, \ \psi_2, \ldots$ where

$$\psi_0 = e^{-\eta}, \tag{69}$$

$$\psi_1 = \frac{1}{2\mathrm{LeN_{bt}}}e^{-2\eta}(h((1 + \mathrm{N_{bt}})\mathrm{N_c} + \mathrm{LeN_{bt}Pr}(-1 + \lambda_1)) + e^{\eta}\mathrm{LeN_{bt}}(-\frac{-2h\mathrm{LeN_{bt}} + h\mathrm{N_c} + h\mathrm{N_{bt}N_c} - h\mathrm{LeN_{bt}Pr} + 3h\mathrm{LeN_{bt}Pr}\lambda_1}{\mathrm{LeN_{bt}}} + h(-2 - 2\eta + \mathrm{Pr}\eta^2 + 2\mathrm{Pr}\lambda_1 + 2\mathrm{Pr}\eta\lambda_1))), \tag{70}$$

$$\psi_2 = \frac{1}{24\mathrm{Le^2}{\mathrm{N_{bt}}}^2}e^{-3\eta}(h^2(4(2 + 3\mathrm{N_{bt}} + {\mathrm{N_{bt}}}^2){\mathrm{N_c}}^2 + 4\mathrm{LeN_{bt}N_c}((3 + 2\mathrm{N_{bt}})\mathrm{Pr} + \mathrm{N_{bt}Sc})(-1 + \lambda_1)\mathrm{Le^2}{\mathrm{N_{bt}}}^2\mathrm{Pr}(1 + 4\mathrm{Pr} - \lambda)(-1 + \lambda_1)^2) + 3e^{2\eta}\mathrm{LeN_{bt}}(4h\mathrm{LeN_{bt}}(-2 + \mathrm{Pr}\eta^2 + 2\mathrm{Pr}\lambda_1 + 2\eta(-1 + \mathrm{Pr}\lambda_1)) - 8\mathrm{LeN_{bt}}(\frac{1}{24\mathrm{Le^2}{\mathrm{N_{bt}}}^2}h^2(4(2 + 3\mathrm{N_{bt}} + {\mathrm{N_{bt}}}^2){\mathrm{N_c}}^2 + 4\mathrm{LeN_{bt}N_c}((3 + 2\mathrm{N_{bt}})\mathrm{Pr} + \mathrm{N_{bt}Sc})(-1 + \lambda_1) + \mathrm{Le^2}{\mathrm{N_{bt}}}^2\mathrm{Pr}(1 + 4\mathrm{Pr} - \lambda)(-1 + \lambda_1)^2) + \frac{1}{2}h(-2 + 2\mathrm{Pr}\lambda_1) + \frac{1}{8\mathrm{Le^2}{\mathrm{N_{bt}}}^2}h(-2h(2 + 3\mathrm{N_{bt}} + {\mathrm{N_{bt}}}^2){\mathrm{N_c}}^2 + \mathrm{LeN_{bt}N_c}(4(1 + \mathrm{N_{bt}}) + h(2 + \mathrm{N_{bt}}(4 + 3\mathrm{Sc} + \mathrm{Pr}(3 - 6\lambda_1)) + \mathrm{Pr}(6 - 6\lambda_1))) + \mathrm{Le^2}{\mathrm{N_{bt}}}^2\mathrm{Pr}(-1 + \lambda_1)(4 + h(15 - 2\lambda + \mathrm{Pr}(1 - 4\lambda_1) + 8\lambda_1 + 2\lambda\lambda_1))) + \frac{1}{8\mathrm{LeN_{bt}}}h^2(-2(1 + \mathrm{N_{bt}})\mathrm{N_c}(-2 + 2\mathrm{Pr}\lambda_1) + \mathrm{LeN_{bt}}(-8 + 2\mathrm{Pr}(-1 + \lambda_1)(7 + \lambda(-1 + \lambda_1) + 3\lambda_1) + \mathrm{Pr}^2(8 + 12\lambda_1 - 4{\lambda_1}^2)))) + h^2(-2(1 + \mathrm{N_{bt}})\mathrm{N_c}(-2 + \mathrm{Pr}\eta^2 + 2\mathrm{Pr}\lambda_1 + 2\eta(-1 + \mathrm{Pr}\lambda_1)) + \mathrm{LeN_{bt}}(4(-2 - 2\eta + \eta) + 2\mathrm{Pr}(-2\eta^3 + \eta^2(2 - 4\lambda_1) + (-1 + \lambda_1)(7 + \lambda(-1 + \lambda_1) + 3\lambda_1) + \eta(-1 + \lambda_1)(7 + \lambda(-1 + \lambda_1) + 3\lambda_1)) + \mathrm{Pr}^2(8 + \eta^4 + 12\lambda_1 + 4\eta^3\lambda_1 - 4{\lambda_1}^2 - 4\eta(-2 - 3\lambda_1 + {\lambda_1}^2) + \eta^2(6 - 2\lambda_1 + 4{\lambda_1}^2)))))) + 3e^{\eta}h(-2h(2 + 3\mathrm{N_{bt}} + {\mathrm{N_{bt}}}^2){\mathrm{N_c}}^2 + \mathrm{Le^2}{\mathrm{N_{bt}}}^2\mathrm{Pr}(-1 + \lambda_1)(4 + h(15 + 2\eta + 2\eta^2 - 2\lambda + 8\lambda_1 + 4\eta\lambda_1 + 2\lambda\lambda_1 + \mathrm{Pr}(1 - 2\eta + 2\eta^2 - 4\lambda_1 + 4\eta\lambda_1))) + \mathrm{LeN_{bt}N_c}(4(1 + \mathrm{N_{bt}}) + h(2 - 12\eta + \mathrm{Pr}(6 + 4\eta^2 - 6\lambda_1 + 8\eta\lambda_1) + \mathrm{N_{bt}}(4 - 8\eta + \mathrm{Sc}(3 + 2\eta + 2\eta^2 + 4\eta\lambda_1) + \mathrm{Pr}(3 + 2\eta^2 - 6\lambda_1 + \eta(-2 + 4\lambda_1))))))), \tag{71}$$

etc.

The HAM series solution for (13) is given by

$$\Phi = \xi_0 + \xi_1 + \xi_2 + \xi_3 + \ldots \tag{72}$$

Using MATHEMATICA, we solve (56) to get $\xi_0, \ \xi_1, \ \xi_2, \ldots$ where

$$\xi_0 = e^{-\eta}, \tag{73}$$

$$\xi_1 = \frac{1}{2\mathrm{N_{bt}}}(e^{-2\eta}(h\mathrm{N_{bt}}\mathrm{Sc}(-1+\lambda_1) + e^{\eta}(2h + 2h\mathrm{N_{bt}} + h\mathrm{N_{bt}}\mathrm{Sc} - 3h\mathrm{N_{bt}}\mathrm{Sc}\lambda_1 + h(-2(1+\eta) + \mathrm{N_{bt}}(-2 - 2\eta + \mathrm{Sc}\eta^2 + 2\mathrm{Sc}\lambda_1 + 2\mathrm{Sc}\eta\lambda_1))))), \tag{74}$$

$$\xi_2 = \frac{1}{24\mathrm{LeN_{bt}}^2}e^{-3\eta}(h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}(1 + 4\mathrm{Sc} - \lambda)(-1+\lambda_1)^2 + 3e^{\eta}h(4\mathrm{LeN_{bt}}^2\mathrm{Sc}(-1+\lambda_1) + h(8(1+\mathrm{N_{bt}})\mathrm{Nc} + \mathrm{LeN_{bt}}(-1\lambda_1)(8\mathrm{Pr} + \mathrm{Sc}(-2 - 4\eta + \mathrm{N_{bt}}(15 + 2\eta + 2\eta^2 - 2\lambda + 8\lambda_1 + 4 + 2\lambda\lambda_1 + \mathrm{Sc}(1 - 2\eta + 2\eta^2 - 4\lambda_1 + 4\eta\lambda_1)))))) + e^{2\eta}(24h\mathrm{LeN_{bt}} + 48h^2\mathrm{LeN_{bt}} + 24h\mathrm{LeN_{bt}}^2 + 24h^2\mathrm{LeN_{bt}}^2 - 36h^2\mathrm{N_c} - 36h^2\mathrm{N_{bt}}\mathrm{Nc} + 36h^2\mathrm{LeN_{bt}}\mathrm{Pr} + 18h^2\mathrm{LeN_{bt}}\mathrm{Sc} + 12h\mathrm{LeN_{bt}}^2\mathrm{Sc} + 86h^2\mathrm{LeN_{bt}}^2\mathrm{Sc} - 25h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}^2 - 11h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}\lambda - 60h^2\mathrm{LeN_{bt}}\mathrm{Pr}\lambda_1 + 6h^2\mathrm{LeN_{bt}}\mathrm{Sc}\lambda_1 - 36h\mathrm{LeN_{bt}}^2\mathrm{Sc}\lambda_1 - 43h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}\lambda_1 - 43h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}^2\lambda_1 + 22h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}\lambda\lambda_1 - 43h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}\lambda 1^2 + 20h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}^2{\lambda_1}^2 - 11h^2\mathrm{LeN_{bt}}^2\mathrm{Sc}\lambda{\lambda_1}^2 + 12h\mathrm{LeN_{bt}}(-2(1+\eta) + \mathrm{N_{bt}}(-2 + \mathrm{Sc}\eta^2 + 2\mathrm{Sc}\lambda_1 + 2\eta(-1 + \mathrm{Sc}\lambda_1))) + h^2(12(1+\mathrm{N_{bt}})\mathrm{Nc}(1+\eta) + \mathrm{LeN_{bt}}(-4(-6(-2 - 2\eta + \eta^2) + \mathrm{Pr}(3 + \eta^3 + \eta(3 - 9\lambda_1) + 3\eta^2(-1+\lambda_1) - 9\lambda_1) + \mathrm{Sc}(6 + 6\eta + 2\eta^3 + 3\eta^2\lambda_1)) + 3\mathrm{N_{bt}}(4(-2 - 2\eta + \eta^2) + 2\mathrm{Sc}(-2\eta^3 + \eta^2(2 - 4\lambda_1) + (-1+\lambda_1)(7 + \lambda(-1+\lambda_1)3\lambda_1) + \eta(-1+\lambda_1)(7 + \lambda(-1+\lambda_1) + 3\lambda_1)) + \mathrm{Sc}^2(8 + \eta^4 + 12\lambda_1 + 4\eta^3\lambda_1 - 4{\lambda_1}^2 - 4\eta(-2 - 3\lambda_1 + {\lambda_1}^2) + \eta^2(6 - 2\lambda_1 + 4{\lambda_1}^2)))))))). \tag{75}$$

The above expressions are the solutions of F, Θ and Φ consists of convergence paremeter h.

4 RESULTS AND DISCUSSION

The solutions for the system of equations (17) - (19) with the boundary conditions are obtained by homotopy analysis method. Using Mathematica equations are solved and the results are shown in the graphs. The convergence of the solution depends on auxiliary linear operator L, auxiliary parameter h and the initial solution. Combined h-curve is plotted to check the convergence of obtained solutions. It can be seen in Figure 1 that the h-curves are horizontal in the ranges $-0.8 < h_F, h_\Theta, h_\Phi < 0.6$.
In Figure 2 we observed that velocity of a nanofluid decreases with increase in non-Newtonian Williamson parameter λ. From Figures 3, 4, 5 it is observed that the temperature decreases with increase in Prandtl number Pr, Lewis number Le and the ratio of Brownian to thermophoretic diffusivities N_{bt}, where as from Figures 6 and 7 we can observed that the temperature increases with increase in λ and Heat capacity ratio N_c. From Figures 8 and 9, it is observed that nanoparticle volume fraction decreases with increase in Schmidt number Sc and N_{bt}. From Figures 10 and 11 we have observed that the heat transfer in fluid increases with increase in N_{bt} and Le. We have obtained radius of convergence for velocity $R = 5.8479$ and for temperature $R = 51.0204$ by drawing Domb-Sykes plot in Figures 12 and 13.

Tables 1 and 2 shows temperature gradient and nanoparticle volume fraction gradient for different parameters. From Table 1, it is observed that the wall temperature gradient decreases with increase in λ and N_c while it increases with increase in Le, N_{bt} and Pr. From Table 2, it can be observed that wall nanoparticle volume fraction gradient decreases with increase in λ and Le while it increases with increase in N_{bt}, Sc and N_c.

5 GRAPHS AND TABLES

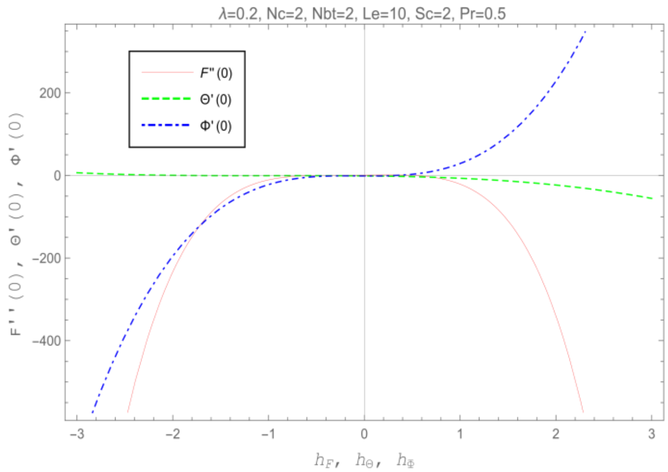

Figure 1: h-curves

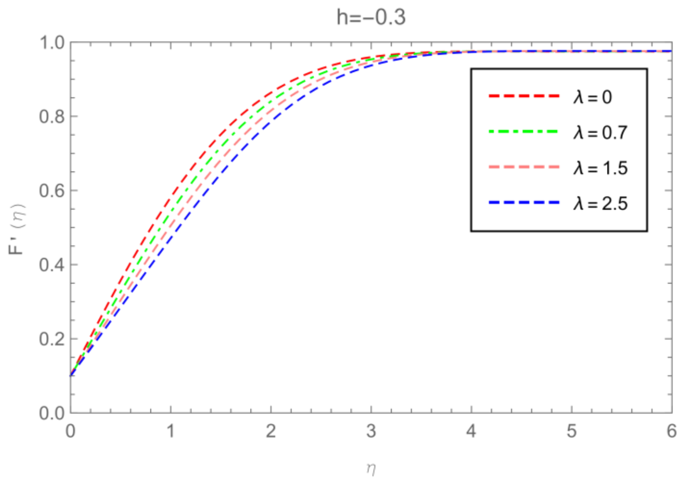

Figure 2: Impact of λ on velocity profile

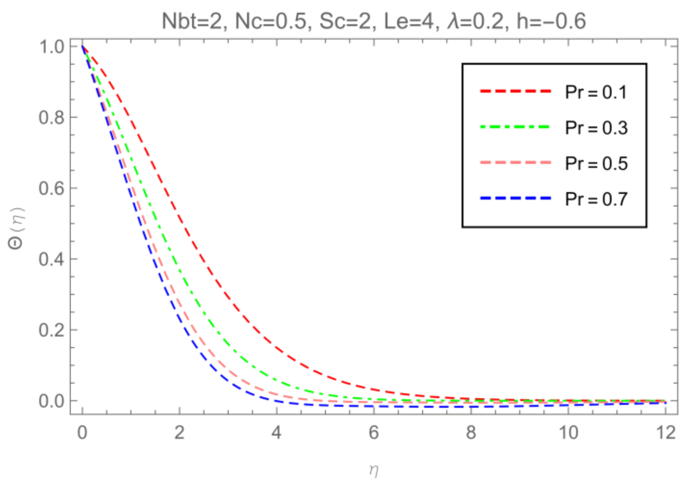

Figure 3: Impact of Pr on temperature profile

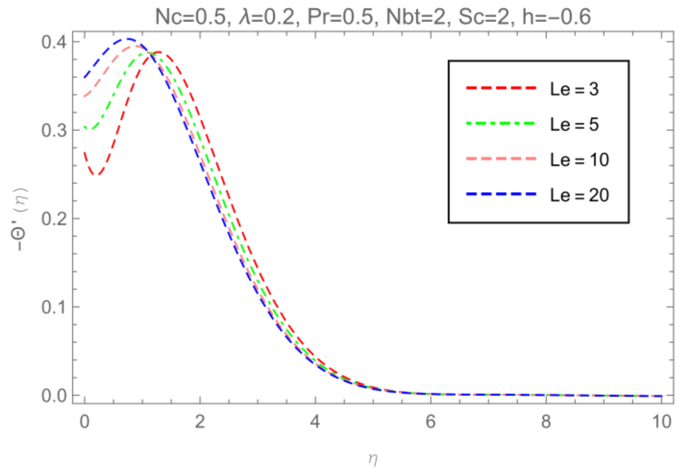

Figure 4: Impact of Le on temperature profile

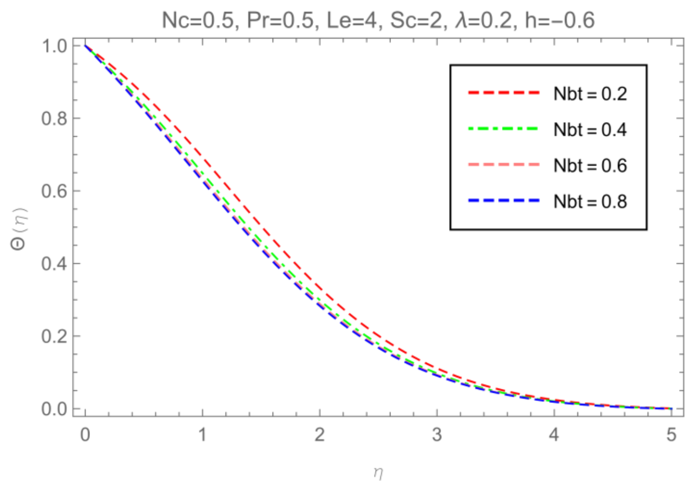

Figure 5: Impact of Nbt on temperature profile

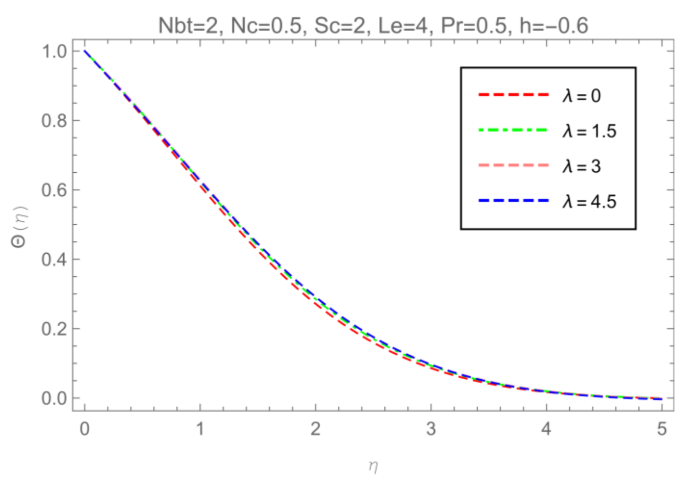

Figure 6: Impact of λ on temperature profile

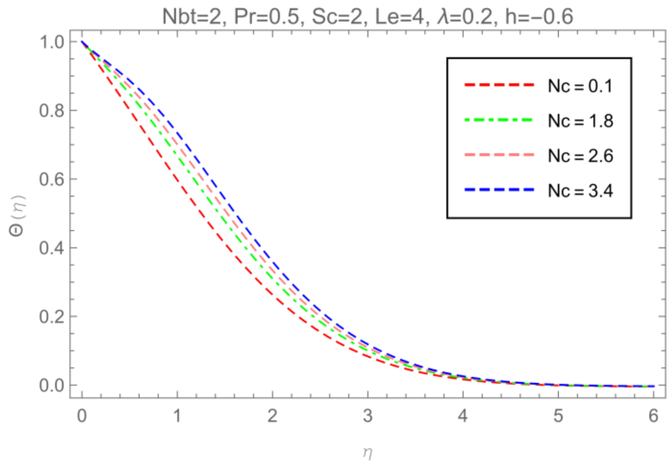

Figure 7: Impact of Nc on temperature profile

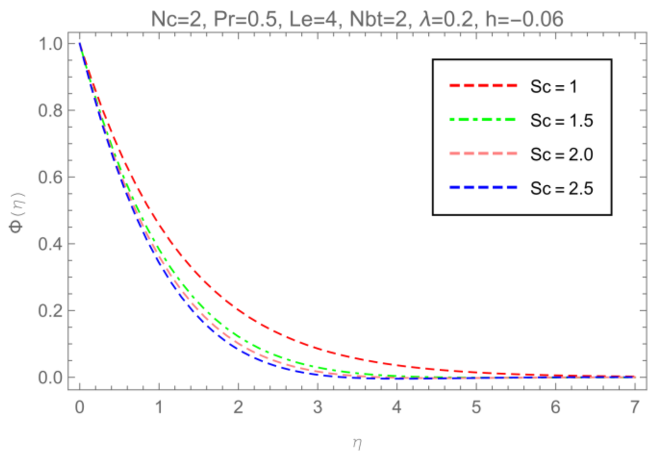

Figure 8: Impact of Sc on concentration profile

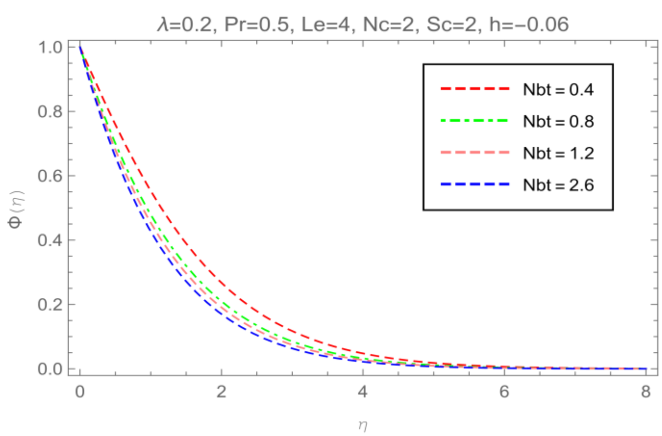

Figure 9: Impact of Nbt on concentration profile

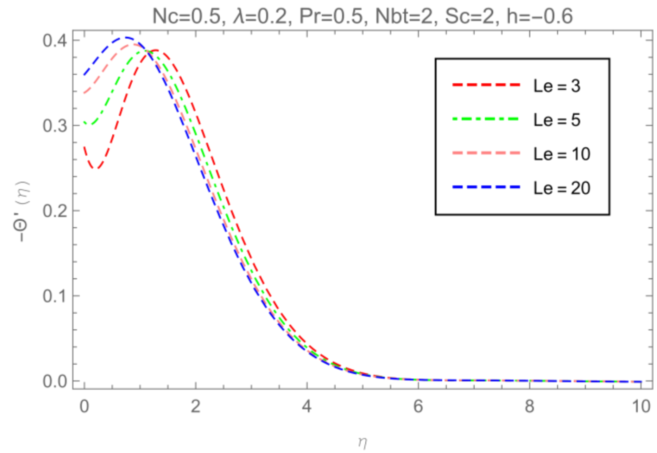

Figure 10: Heat transfer in fluid against different values of Le

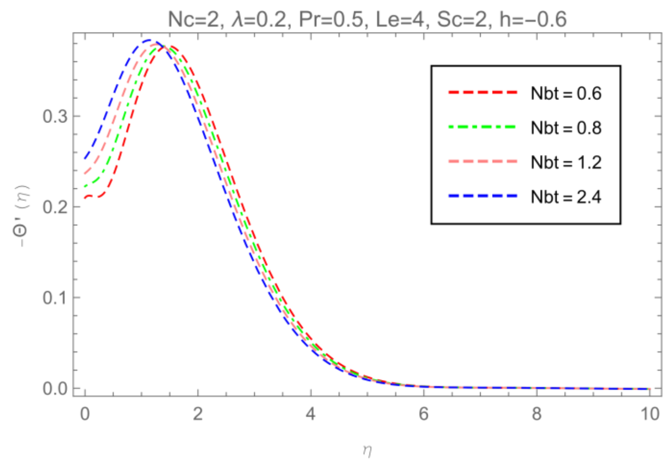

Figure 11: Heat transfer in fluid against different values of Nbt

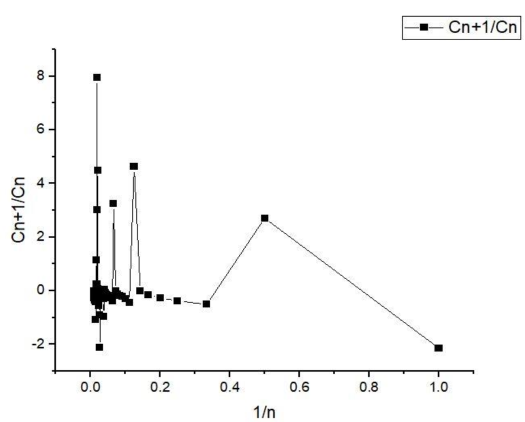

Figure 12. Domb-Sykes Velocity Plot with $R = 51.0204$

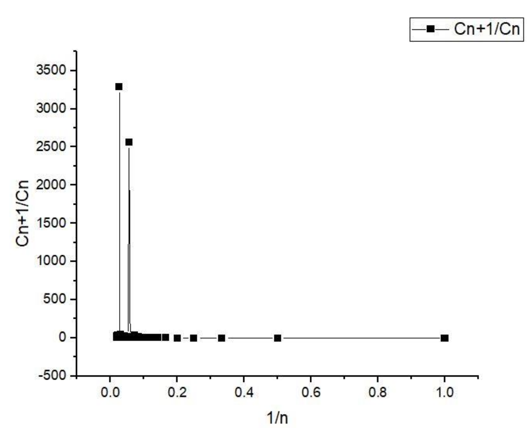

Figure 13. Domb-Sykes Temperature Plot with $R = 5.8479$

Table 1: Values of temperature gradient $-\Theta'(0)$ when $h = -0.6$

λ	Le	N_{bt}	N_c	Pr	$-\Theta'(0)$
0	4	2	0.5	0.5	0.357032
0.2					0.35425
0.4					0.35173
0.2	4	2	0.5	0.5	0.35425
	10				0.371597
	20				0.377733
0.2	4	0.5	0.5	0.5	0.310047
		1			0.339205
		2			0.35425
0.2	4	2	0.5	0.5	0.35425
			1		0.328814
			2		0.290548
0.2	4	2	0.5	0.2	0.214633
				0.6	0.397238
				0.8	0.51975

Table 2: Values of nanoparticle volume fraction gradient $-\Phi'(0)$ when $h = -0.06$

λ	Le	N_{bt}	S_c	N_c	$-\Phi'(0)$
0	4	2	0.5	0.5	0.799672
0.2					0.799604
0.4					0.799535
0.2	4	2	0.5	0.5	0.799604
	10				0.799318
	20				0.799222
0.2	4	0.5	0.5	0.5	0.566004
		1			0.721423
		2			0.799604
0.2	4	2	0.5	0.5	0.799604
			1		0.84357
			2		0.918342
0.2	4	2	0.5	0.5	0.799604
				1	0.80008
				2	0.801029

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