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Approximation by Generalized λ-Kantorovich Operators

Sangeeta Garg

Department of Science, Mewar Institute of Management (CCS University), Vasundhara, Ghaziabad (Delhi-NCR), India.

Email : [1] sangeetavipin7@gmail.com

***Abstract* -** In this paper we deal with λ-Kantorovich operators which are to be generalized. These λ-Kantorovich operators are some modification of Bernstein operators which depends on parameter $\lambda \in [-1,1]$. We consider here the Stancu type generalization of these type of linear positive operators, λ-Kantorovich operators. We obtain their moments and prove Voronovskaya type theorem as well as Grũss-Voronovskaya type theorem for such operators.

Keywords: *Bernstein operators; λ-Kantorovich operators; Peetre's K-functional; Stancu type generalization; Voronovskaya Theorems.*

2020 Mathematics Subject Classification: *41A10, 41A25*, 41*A*36.

INTRODUCTION

In the year 1912, Bernstein [6] introduced the Bernstein polynomials in order to prove Weierstrass first fundamental theorem. The Bernstein polynomials have too notable approximation properties to make an area of intensive research. Referring [13]-[16] etc. the Bernstein polynomials of order n are given by

$$B_n(f,x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $x \in [0,1]$ and $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Recently CAI Et Al. [7] introduced and considered a new generalization of the Bernstein polynomials depending on the parameter λ as follows:

$$B_{n,\lambda}(f,x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda,x) f\left(\frac{k}{n}\right)$$

where $\lambda \in [-1,1]$ and

- $\tilde{b}_{n,0}(\lambda,x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x)$
- $\tilde{b}_{n,k}(\lambda,x) = b_{n,k}(x) + \lambda \left\{\frac{n-2k+1}{n^2-1}\right\} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(\lambda,x)$
- $\tilde{b}_{n,n}(\lambda,x) = b_{n,n}(\lambda,x) - \frac{\lambda}{n+1} b_{n+1,n}(\lambda,x)$

Particularly, if $\lambda = 0$ then λ-Bernstein operators reduce to the well-known Bernstein operators. Acu et al. [2],[3] have deeply studied many interesting approximation properties of the λ-Bernstein operators such as uniform convergence, rate of convergence in terms of modulus of continuity, Voronovskaya type pointwise convergence and shape preserving properties.

The classical Kantorovich operators [17] are the integral modification of Bernstein operators so as to approximate integrable functions defined on $[0,1]$. These operators and several other Kantorovich variants attracted the interest of number of authors viz. Özarslan-Duman [19], Dhamija-Deo [8], Acu-Rasa [4] and

Gupta [12] etc. Now the Kantorovich variant of λ-Bernstein operators are

$$K_{n,\lambda}(f,x) = (n+1)\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda,x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \qquad \ldots.. \qquad (1.1)$$

In 1983, D.D. Stancu [20] introduced an important generalization of operators for $0 \le \alpha \le \beta$. Our aim is to consider the Stancu type generalization of operators (1.1) which make our study more general. Several authors [5], [11] have considered Stancu type generalization for various operators and obtain valuable results. Stancu type generalization of operators (1.1) for $0 \le \alpha \le \beta$, can be represented as

$$K_{n,\lambda}^{\alpha,\beta}(f,x) = (n+1)\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda,x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f\left(\frac{nt+\alpha}{n+\beta}\right)dt. \qquad \ldots \qquad (1.2)$$

From here we see that $\alpha = \beta = 0$, operators (1.2) yield operators (1.1). In this paper, we discuss convergence properties and Voronovskaya type theorems for operators (1.2). In this paper we provide Grüss-Voronovskaya type theorem [1] for λ-Kantorovich Stancu operators (1.2).

BASIC RESULTS

In this section, we give some results related to our operators. These play important role to find main results.

Lemma-1 For $0 \le \alpha \le \beta$, if we define

$$K_{n,\lambda}^{\alpha,\beta}(e_0,x) = 1$$

$$K_{n,\lambda}^{\alpha,\beta}(e_1,x) = \left(\frac{n}{n+\beta}\right)\left\{\frac{2nx+1}{2(n+1)} + \frac{\alpha}{n} + \lambda\cdot\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\right\}$$

$$K_{n,\lambda}^{\alpha,\beta}(e_2,x) = \left(\frac{n}{n+\beta}\right)^2\left[\frac{3n(n-1)x^2-6nx-1}{3(n+1)^2} + \frac{\alpha}{n}\cdot\frac{2nx+1}{(n+1)} + \frac{\alpha^2}{n^2} + 2n\alpha\lambda\left\{\frac{(n-1)x-2nx^2}{\alpha(n+1)^2(n-1)} + \frac{1-2x-(1-x)^{n+1}}{n(n^2-1)} + \frac{n(n+1)+\alpha}{n\alpha(n^2-1)}x^{n+1}\right\}\right]$$

$$K_{n,\lambda}^{\alpha,\beta}(e_3,x) = \left(\frac{n}{n+\beta}\right)^3\left\{x^3 + \frac{1+14nx+18n(n-1)x^2-4n(6n+1)x^3}{4(n+1)^3}\right\} + \frac{3n\alpha}{(n+\beta)^3}\left\{nx^2 - \frac{n}{3}\frac{3(3n+1)x^2-6nx-1}{(n+1)^2} + \alpha\frac{2nx-1}{2(n+1)} + \frac{\alpha^2}{3n}\right\} + \frac{3n^3\alpha\lambda}{(n+\beta)^3}\left\{\frac{1-2(3n-4)x+6n(n-5)x^2-12n(n-1)x^3}{6\alpha(n+1)^3(n-1)-(1-x)^{n+1}} + \frac{(6n^2+12n+7)x^{n+1}}{6\alpha(n+1)^3(n-1)} + 2\frac{(n-1)x-2nx^2+(n+1)x^{n+1}}{n(n+1)^2(n-1)} + \alpha\times\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2(n^2-1)}\right\}.$$

The proof of the above lemma follows along the lines of Acu-Rasa [4].

Lemma2. For $0 \le \alpha \le \beta$, central moments for $K_{n,\lambda}^{\alpha,\beta}(f(t),x)$ are defined by

$$
\begin{aligned}
&K_{n,\lambda}^{\alpha,\beta}(e_1-e_0x,x)\\
=\ &\left(\frac{n}{n+\beta}\right)K_{n,\lambda}(e_1,x)+\left(\frac{\alpha}{n+\beta}-x\right)K_{n,\lambda}(e_0,x)\\
=\ &\left(\frac{n}{n+\beta}\right)\left\{\frac{2nx+1}{2(n+1)}+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda\right\}\\
&+\left(\frac{\alpha}{n+\beta}-x\right)\\
&K_{n,\lambda}^{\alpha,\beta}((e_1-e_0x)^2,x)\\
=\ &\left(\frac{n}{n+\beta}\right)^2K_{n,\lambda}(e_2,x)-\left(\frac{2n}{n+\beta}\right)\left(\frac{\alpha}{n+\beta}-x\right)K_{n,\lambda}(e_1,x)\\
&+\left(\frac{\alpha}{n+\beta}-x\right)^2K_{n,\lambda}^{\alpha,\beta}(e_0,x)\\
=\ &\left(\frac{n}{n+\beta}\right)^2\left[\frac{3n(n-1)x^2-6nx-1}{3(n+1)^2}+2\lambda\cdot\left\{\frac{(n-1)x-2nx^2}{(n+1)^2(n-1)}\right.\right.\\
&\qquad\left.\left.+\frac{(n+1)x^{n+1}}{n^2-1}\right\}\right]-\left(\frac{2n}{n+\beta}\right)\left(\frac{\alpha}{n+\beta}-x\right)\left\{\frac{2nx+1}{2(n+1)}\right.\\
&\qquad\left.+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda\right\}+\left(\frac{\alpha}{n+\beta}-x\right)^2.
\end{aligned}
$$

Lemma3. Central moments of λ-Kantorovich operators yield

$$
\begin{aligned}
\left|K_{n,\lambda}^{\alpha,\beta}(e_1-e_0x,x)\right| &\le \sigma_{n,\alpha,\beta}^{\lambda}\\
\left|K_{n,\lambda}^{\alpha,\beta}((e_1-e_0x)^2,x)\right| &\le \rho_{n,\alpha,\beta}^{\lambda},
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{n,\alpha,\beta}^{\lambda} &=\left(\frac{n}{n+\beta}\right)\left\{\frac{1}{2(n+1)}+\frac{\alpha}{n}+\frac{\lambda}{n^2-1}\right\},\\
\rho_{n,\alpha,\beta}^{\lambda} &=\left(\frac{n}{n+\beta}\right)^2\left[\frac{1}{3(n+1)^2}+\frac{\alpha}{n(n+1)}-\left(\frac{\alpha}{n}\right)^2+\frac{2\alpha\lambda}{n(n^2-1)}\right].
\end{aligned}
$$

Lemma4. The λ-Kantorovich operators yield too

1) $\lim_{\to\infty}K_{n,\lambda}^{\alpha,\beta}(e_1-e_0x,x)=0$
2) $\lim_{n\to\infty}K_{n,\lambda}^{\alpha,\beta}((e_1-e_0x)^2,x)=4x^2.$

CONVERGENCE PROPERTIES OF $\mathbf{K_{n,\lambda}^{\alpha,\beta}}$

In this section, we investigate some approximation properties of convergence for the mentioned operators and find the rate of convergence by using moduli of continuity.

Theorem1. For $f\in C[0,1]$, there exists

$$\lim_{n\to\infty}K_{n,\lambda}^{\alpha,\beta}(f,x)=f(x)\qquad \text{uniformly on } [0,1].$$

Proof: It is obvious from Lemma 2 that

$$\lim_{n\to\infty}K_{n,\lambda}^{\alpha,\beta}(e_k,x)=e_k$$

on [0,1] uniformly for $k=0,1,2,3$. Hence the Bohmann-Korovkin [18] theorem states required proof.

Theorem2. For $g\in C[0,1]$, we have

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right|\le 2\omega\left(g,\sqrt{\rho_{n,\alpha,\beta}^{\lambda}}\right),$$

where ω represents the modulus of continuity.

Proof: Applying the general property of modulus of continuity

$$|g(t)-g(x)| \leq \omega(g,\delta)\left|\frac{(t-x)^2}{\delta^2}+1\right|$$

we obtain

$$\begin{aligned}\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right| &\leq K_{n,\lambda}^{\alpha,\beta}(|g(t)-g(x)|,x) \\ &\leq \omega(g,\delta)\left(1+\frac{1}{\delta^2}K_{n,\lambda}^{\alpha,\beta}((t-x)^2,x)\right)\end{aligned}$$

choosing $\delta=\sqrt{\rho_{n,\alpha,\beta}^{\lambda}}$ and then applying lemma 4, we have the required result.

Theorem3. Let $g \in C^1[0,1]$, we can have

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right| \leq \sigma_{n,\alpha,\beta}^{\lambda}|g'(x)|+2\omega\left(g',\sqrt{\rho_{n,\alpha,\beta}^{\lambda}}\right)\sqrt{\rho_{n,\alpha,\beta}^{\lambda}}.$$

Proof. Since $g \in C^1[0,1]$, for $x,t \in [0,1]$ we have

$$g(t)-g(x)=g'(x)(t-x)+\int_x^t \{g'(y)-g'(x)\}dy.$$

Operating both sides by $K_{n,\lambda}^{\alpha,\beta}$, we get

$$\begin{aligned}K_{n,\lambda}^{\alpha,\beta}(\{g(t)-g(x)\},x)= \quad & g'(x)K_{n,\lambda}^{\alpha,\beta}(t-x,x)+ \\ & K_{n,\lambda}^{\alpha,\beta}\left\{\int_x^t \{g'(y)-g'(x)\}dy,x\right\}.\end{aligned}$$

Now using well-known property of modulus of continuity, we find

$$\left|K_{n,\lambda}^{\alpha,\beta}\left\{\int_x^t\{g'(y)-g'(x)\}dy,x\right\}\right| \leq \omega(g',\delta)\left[\frac{(t-x)^2}{\delta}+|t-x|\right],\ \delta>0$$

Therefore

$$\begin{aligned}\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right| \leq \quad & |g'(x)|\left|K_{n,\lambda}^{\alpha,\beta}(t-x,x)\right|+\omega(g',\delta)\times \\ & \left\{\frac{1}{\delta}K_{n,\lambda}^{\alpha,\beta}((t-x)^2,x)+K_{n,\lambda}^{\alpha,\beta}(|t-x|,x)\right\}.\end{aligned}$$

Applying Cauchy-Schwarz inequality and then from lemma 4

$$\begin{aligned} & \left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right| \\ \leq \quad & |g'(x)|\left|K_{n,\lambda}^{\alpha,\beta}(t-x,x)\right|+\omega(g',\delta)\left\{\frac{1}{\delta}\sqrt{K_{n,\lambda}^{\alpha,\beta}((t-x)^2,x)}+1\right\}\times \\ & \sqrt{K_{n,\lambda}^{\alpha,\beta}((t-x)^2,x)} \\ \leq \quad & |g'(x)|\sigma_{n,\alpha,\beta}^{\lambda}+\omega(g',\delta)\left\{\frac{1}{\delta}\sqrt{\rho_{n,\alpha,\beta}^{\lambda}}+1\right\}\sqrt{\rho_{n,\alpha,\beta}^{\lambda}}.\end{aligned}$$

Choosing $\delta=\sqrt{\rho_{n,\alpha,\beta}^{\lambda}}$, we have the desired result. Definition-1 (Peetre's K-functional) Let us consider space $C_B[0,\infty)$ consisting of all those functions f which are continuous as well as bounded and possess the norm $\| f \|=\sup_{x\in[0,\infty)}|f(x)|$. Then Peetre's K-functional for $\delta>0$, is defined by

$$K_2(f,\delta)=\inf_{g\in W^2}\{\| f-g \|+\delta\|g''\|\},$$

where $W^2 = \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$ for $\delta \geq 0$ and $C > 0$. Also $\exists$ an absolutely constant C in such a way that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta})$$

where $\omega_2(f,\sqrt{\delta})$ represents continuous module of f in $C_B[0,\infty)$ of second order and

$$\omega_2(f,\sqrt{\delta}) = \sup_{h\in(0,\sqrt{\delta}),x\in[0,\infty)} |f_{x+2h} - 2f_{x+h} + f_x|$$

Theorem 4. If $g \in C[0,1]$, then for a constant $M > 0$, we have

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x) - g(x)\right| \leq M\omega_2\left(f, \frac{1}{2}\sqrt{\rho_{n,\alpha,\beta}^{\lambda} + \left(\sigma_{n,\alpha,\beta}^{\lambda}\right)^2}\right) + \omega\left(g', \sigma_{n,\alpha,\beta}^{\lambda}\right).$$

Proof. Let us denote

$$\theta_{n,\lambda}^{\alpha,\beta}(x) = \frac{n}{n+\beta}\left\{\frac{2nx+1}{2(n+1)} + \frac{\alpha}{n} + \lambda.\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\right\}$$

and

$$\tilde{K}_{n,\lambda}^{\alpha,\beta}(g,x) = K_{n,\lambda}^{\alpha,\beta}(g,x) + g(x) - g\left(\theta_{n,\lambda}^{\alpha,\beta}(x)\right). \quad \ldots \quad (3.1)$$

This provides us

$$\tilde{K}_{n,\lambda}^{\alpha,\beta}(e_0,x) = K_{n,\lambda}^{\alpha,\beta}(e_0,x) = 1$$
$$\tilde{K}_{n,\lambda}^{\alpha,\beta}(e_1,x) = K_{n,\lambda}^{\alpha,\beta}(e_1,x) + x - \theta_{n,\lambda}^{\alpha,\beta}(x) = x$$

Now, we know Taylor's formula for some function h as

$$h(t) = h(x) + \int_x^t (t-y)h''(y)dy$$

Therefore operating by the operators $\tilde{K}_{n,\lambda}^{\alpha,\beta}$ on both sides

$$\begin{aligned}\tilde{K}_{n,\lambda}^{\alpha,\beta}(h,x) &= h(x) + \tilde{K}_{n,\lambda}^{\alpha,\beta}\left(\int_x^t (t-y)h''(y)dy, x\right)\\ &= h(x) + K_{n,\lambda}^{\alpha,\beta}\left(\int_x^t (t-y)h''(y)dy, x\right) - \\ &\quad \int_x^{\theta_{n,\lambda}^{\alpha,\beta}(x)} \left\{\theta_{n,\lambda}^{\alpha,\beta}(x) - y\right\} h''(y)dy.\end{aligned}$$

From here we can have

$$\begin{aligned}\left|\tilde{K}_{n,\lambda}^{\alpha,\beta}(h,x) - h(x)\right| &\leq \left|K_{n,\lambda}^{\alpha,\beta}\left(\int_x^t (t-y)h''(y)dy, x\right)\right| + \left|\int_x^{\theta_{n,\lambda}^{\alpha,\beta}(x)} \left\{\theta_{n,\lambda}^{\alpha,\beta}(x) - y\right\} h''(y)dy\right|\\ &\leq K_{n,\lambda}^{\alpha,\beta}((t-x)^2, x)\|h''\| + \left\{\theta_{n,\lambda}^{\alpha,\beta}(x) - x\right\}^2 \|h''\|\\ &\leq \left\{\rho_{n,\alpha,\beta}^{\lambda} + \left(\sigma_{n,\alpha,\beta}^{\lambda}\right)^2\right\}\|h''\|.\end{aligned}$$

In order to equation (3.1), we have

$$\left|\tilde{K}_{n,\lambda}^{\alpha,\beta}(g,x)\right| \leq \left|K_{n,\lambda}^{\alpha,\beta}(g,x)\right| + |g(x)| + \left|g\left(\theta_{n,\lambda}^{\alpha,\beta}(x)\right)\right| \leq 3 \| g \| \quad \ldots \quad (3.2)$$

Now for $g \in C[0,1]$ and $h \in W^2[0,1]$, using equations (3.1) and (3.2) we obtain

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right| = \left|\tilde{K}_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)+g\left(\theta_{n,\lambda}^{\alpha,\beta}(x)\right)-g(x)\right|$$
$$\leq \mid \tilde{K}_{n,\lambda}^{\alpha,\beta}\left(g-h,x\right|+|\tilde{K}_{n,\lambda}^{\alpha,\beta}(h,x)-h(x)\mid +$$
$$|h(x)-g(x)|+\left|g\left(\theta_{n,\lambda}^{\alpha,\beta}(x)\right)-g(x)\right|$$
$$\leq 4 \parallel g-h \parallel +\{ \quad \rho_{n,\alpha,\beta}^{\lambda}+\left(\sigma_{n,\alpha,\beta}^{\lambda}\right)^2\right\} \|h''\| + \omega\left(g,\sigma_{n,\alpha,\beta}^{\lambda}\right).$$

Taking the infimum on the RHS overall $h \in W^2[0,1]$, we have

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right| \leq 4K_2\left(g,\frac{\rho_{n,\alpha,\beta}^{\lambda}+\left(\sigma_{n,\alpha,\beta}^{\lambda}\right)^2}{4}\right)+\omega\left(g,\sigma_{n,\alpha,\beta}^{\lambda}\right).$$

Finally using equivalence between Peetre's K-functional and second order modulus of continuity

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)\right| \leq M\omega_2\left(f,\frac{1}{2}\sqrt{\rho_{n,\alpha,\beta}^{\lambda}+\left(\sigma_{n,\alpha,\beta}^{\lambda}\right)^2}\right)+\omega\left(g,\sigma_{n,\alpha,\beta}^{\lambda}\right)$$

and hence we get the proof of theorem.

Definition2. (Ditzian-Totik modulus of smoothness) [9] Let $g \in C[0,1]$ and $\phi(x)=\sqrt{x(1-x)}$ then Ditzian-Totik modulus of smoothness of first order is defined as

$$\omega_{\phi}(g,t)$$
$$= \sup_{0<h\leq t}\left[\left|g\left(x+\frac{h\phi(x)}{2}\right)-g\left(x-\frac{h\phi(x)}{2}\right)\right|, x \pm \frac{h\phi(x)}{2} \in [0,1]\right].$$

The corresponding K-functional of this Ditzian-Totik smoothness is given by

$$K_{\phi}(g,t)=\inf_{h\in W_{\phi}[0,1]}\{\parallel g-h \parallel +t\|\phi h'\|, t>0\} \quad \ldots \quad (3.3)$$

where $W_{\phi}[0,1]=\{h: h \in AC_{\text{loc}}[0,1], \phi h' < \infty\}$. $AC_{loc}[0,1]$ is the class of absolutely continuous functions on each $[a,b] \subset [0,1]$. The equivalance relation between K-functional and Ditzian-Totik smoothness of first order is

$$K_{\phi}(g,t) \leq \omega_{\phi}(g,t),\ (C>0). \quad \ldots. \quad (3.4)$$

Theorem 5. For some $g \in C^2[0,1]$ and sufficiently large n, there holds an inequality

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)-A_n(x,\lambda)g'(x)-B_n(x,\lambda)g''(x)\right|$$
$$\leq C\phi^2(x)\omega_{\phi}\left(g'',\sqrt{n}\right),$$

where $C>0$ and

$$A_n^{\alpha,\beta}(x,\lambda) = K_{n,\lambda}^{\alpha,\beta}(e_1-e_0x,x)$$
$$B_n^{\alpha,\beta}(x,\lambda) = \frac{1}{2}K_{n,\lambda}^{\alpha,\beta}((e_1-e_0x)^2,x)$$

Proof. For some $g \in C^2[0,1]$ and $t,x \in [0,1]$, Taylor's expansion states that

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t (t-y)g''(y)dy$$

From here we can have

$$g(t)-g(x)-(t-x)g'(x)-\frac{1}{2}(t-x)^2g''(x)$$
$$= \int_x^t (t-y)g''(y)dy-\int_x^t (t-y)g''(x)dy$$
$$= \int_x^t (t-y)\{g''(y)-g''(x)\}dy$$

Operating with $K_{n,\lambda}^{\alpha,\beta}$ on both sides, we get

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)-A_n^{\alpha,\beta}(x,\lambda)g'(x)-B_n^{\alpha,\beta}(x,\lambda)g''(x)\right|$$
$$\leq K_{n,\lambda}^{\alpha,\beta}\left(\left|\int_x^t\right| t-y \parallel g''(y)-g''(x)|dy|,x\right) \quad \ldots \quad .(3.5)$$

$\leq K_{n,\lambda}^{\alpha,\beta}(2\|g''-h\|(t-x)^2+2\|\phi h'\|\phi^{-1}(x)|t-$
$y|^3,x)$ .... (3.6)

according to Finta [10], where $h\in W_\phi[0,1]$. Now for n to be sufficiently large, using Lemma 4

$$K_{n,\lambda}^{\alpha,\beta}((e_1-e_0x)^2,x)\leq C_1\phi^2(x),\ K_{n,\lambda}^{\alpha,\beta}(g,x)\leq C_2\phi^4(x) \quad \ldots \quad (3.7)$$

From (3.5)-(3.7) and Cauchy Schwarz inequality, we obtain

$$\left|K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)-A_n^{\alpha,\beta}(x,\lambda)g'(x)-B_n^{\alpha,\beta}(x,\lambda)g''(x)\right|$$
$$\leq 2\|g''-h\|K_{n,\lambda}^{\alpha,\beta}((t-x)^2,x)+2\|\phi h'\|\phi^{-1}(x)K_{n,\lambda}^{\alpha,\beta}(|t-x|^3,x)$$
$$\leq 2\|g''-h\|C_1\phi^2(x)+2\|\phi h'\|\phi^{-1}(x)\left\{K_{n,\lambda}^{\alpha,\beta}((e_1-e_0x)^2,x)\right\}^{1/2}\times$$
$$\left\{K_{n,\lambda}^{\alpha,\beta}((e_1-e_0x)^4,x)\right\}^{1/2}$$
$$\leq 2C_1\|g''-h\|\phi^2(x)+2\|\phi h'\|\phi^{-1}(x)\cdot C_1\phi(x)C_2\phi^2(x)$$
$$\leq C\phi^2(x)\{\|g''-h\|+C_2\|\phi h'\|\}.\ C=2C_1$$

Choosing $C_2=\sqrt{n}$ and taking infimum on RHS over $h\in W_\phi[0,1]$, Peetre's K-functional gives the required result.

Corollary1. For $g\in C^2[0,1]$

$$\lim_{n\to\infty}\left[K_{n,\lambda}^{\alpha,\beta}(g,x)-g(x)-A_n^{\alpha,\beta}(x,\lambda)g'(x)-B_n^{\alpha,\beta}(x,\lambda)g''(x)\right]=0$$

Its proof is obvious.

Theorem6. For $f,g\in C^2[0,1]$ and $x\in[0,1]$,

$$\lim_{n\to\infty}\left[K_{n,\lambda}^{\alpha,\beta}(fg,x)-K_{n,\lambda}^{\alpha,\beta}(f,x)K_{n,\lambda}^{\alpha,\beta}(g,x)\right]=8f'(x)g'(x)x^2.$$

Proof. We have

$$
\begin{aligned}
& K_{n,\lambda}^{\alpha,\beta}(fg,x) - K_{n,\lambda}^{\alpha,\beta}(f,x)K_{n,\lambda}^{\alpha,\beta}(g,x) \\
= \; & K_{n,\lambda}^{\alpha,\beta}(fg,x) - f(x)g(x) - (fg)' A_n^{\alpha,\beta}(x,\lambda) - (fg)'' B_n^{\alpha,\beta}(x,\lambda) - \\
& g(x)\left[K_{n,\lambda}^{\alpha,\beta}(f,x) - f(x) - A_n^{\alpha,\beta}(x,\lambda)f'(x) - B_n^{\alpha,\beta}(x,\lambda)f''(x)\right] - \\
& K_{n,\lambda}^{\alpha,\beta}(f,x)\left[K_{n,\lambda}^{\alpha,\beta}(g,x) - g(x) - A_n^{\alpha,\beta}(x,\lambda)g'(x) - B_n^{\alpha,\beta}(x,\lambda) \times \right. \\
& \left. g''(x)\right] + B_n^{\alpha,\beta}(x,\lambda)\left[f(x)g''(x) + 2f'(x)g'(x) - g'(x)K_{n,\lambda}^{\alpha,\beta}(f,x)\right] \\
& + A_n^{\alpha,\beta}(x,\lambda)\left[f(x)g'(x) - g'(x)K_{n,\lambda}^{\alpha,\beta}(f,x)\right]
\end{aligned}
$$

Using Theorem 3, Corollary 1 and Theorem 4, we obtain

$$
\begin{aligned}
& \lim_{n\to\infty}\left[K_{n,\lambda}^{\alpha,\beta}(fg,x) - K_{n,\lambda}^{\alpha,\beta}(f,x)K_{n,\lambda}^{\alpha,\beta}(g,x)\right] \\
= \; & \lim_{n\to\infty} 2f'(x)g'(x)B_n^{\alpha,\beta}(x,\lambda) + \lim_{n\to\infty} g''(x)\left[f(x) - K_{n,\lambda}^{\alpha,\beta}(f,x)\right] \\
& \times B_n^{\alpha,\beta}(x,\lambda) + \lim_{n\to\infty} g'(x)\left[f(x) - K_{n,\lambda}^{\alpha,\beta}(f,x)\right] A_n^{\alpha,\beta}(x,\lambda) \\
= \; & 8f'(x)g'(x)x^2.
\end{aligned}
$$

Hence the proof of theorem as required.

CONCLUSION

By all counts and with proven results, it is no wonder to say that our operators considered in this research article are very compatible to the discipline of approximation theory. Results and proof of main theorem are very precisely explained. Eventually, we may conclude that this research paper is explicit.

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